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Rainbow game domination subdivision number of a graph

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Abstract

The rainbow game domination subdivision number of a graph G is defined by the following game. Two players $\mathcal{D}$ and $\mathcal{A}$, $\mathcal{D}$ playing first, alternately mark or subdivide an edge of G which is not yet marked nor subdivided. The game ends when all the edges of G are marked or subdivided and results in a new graph G'. The purpose of $\mathcal{D}$ is to minimize the 2-rainbow dominating number $\gamma_{r2}(G')$ of G' while $\mathcal{A}$ tries to maximize it. If both $\mathcal{A}$ and $\mathcal{D}$ play according to their optimal strategies, $\gamma_{r2}(G')$ is well defined. We call this number the *rainbow game domination subdivision number* of G and denote it by $\gamma_{rg}(G)$.

In this paper we initiate the study of the rainbow game domination subdivision number of a graph and present sharp bounds on the rainbow game domination subdivision number of a tree.

Keywords: rainbow domination number, rainbow game domination subdivision number
MSC 2010: 05C69, 05C05

1 Introduction

In this paper, G is a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly V and E). The number of vertices of a graph G is its *order* $n = n(G)$. For every vertex $v \in V$, the *open neighborhood* $N_G(v) = N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. A *subdivision* of an edge uv is obtained by removing the edge uv, adding a new vertex w, and adding edges uw and wv. A vertex of degree one is a *leaf* and a *support vertex* is a vertex that is adjacent to at least one leaf. A vertex $v \in V$ is said to *dominate* all the vertices in its closed neighborhood $N[v]$. A subset D of V is a *dominating set* of G if D dominates every vertex of $V \setminus D$ at least once. The *domination number* $\gamma(G)$ is the minimum cardinality among all dominating sets of G. Similarly, a subset D of V is

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a 2-*dominating set* of G if D dominates every vertex of $V \setminus D$ at least twice. The 2-*domination number* $\gamma_{r2}(G)$ is the minimum cardinality among all 2-dominating sets of G. We refer the reader to the books [7, 10] for graph theory notation and terminology not defined here.

The game domination subdivision number of graph G, introduced by Favaron et al. in [6], is defined by the following game. Two players $\mathcal{A}$ and $\mathcal{D}$ alternately play on a given graph G, $\mathcal{D}$ playing first, by marking or subdividing an edge of G. An edge which is neither marked nor subdivided is said to be *free*. At the beginning of the game, all the edges of G are free. At each turn, $\mathcal{D}$ marks a free edge of G and $\mathcal{A}$ subdivides a free edge of G by a new vertex. The game ends when all the edges of G are marked or subdivided and results in a new graph G'. The purpose of $\mathcal{D}$ is to minimize the domination number $\gamma(G')$ of G' while $\mathcal{A}$ tries to maximize it. If both $\mathcal{A}$ and $\mathcal{D}$ play according to their optimal strategies, $\gamma(G')$ is well defined. This number, denoted by $\gamma_{gs}(G)$, is called the *game domination subdivision number* of G.

For a positive integer k, a *k-rainbow dominating function* (kRDF) of a graph G is a function f from the vertex set $V(G)$ to the set of all subsets of the set $\{1, 2, \ldots, k\}$ such that for any vertex $v \in V(G)$ with $f(v) = \emptyset$ the condition $\bigcup_{u \in N(v)} f(u) = \{1, 2, \ldots, k\}$ is fulfilled. The *weight* of a kRDF f is the value $\omega(f) = \sum_{v \in V} |f(v)|$. The *$k$-rainbow domination number* of a graph G, denoted by $\gamma_{rk}(G)$, is the minimum weight of a kRDF of G. A *$\gamma_{rk}(G)$-function* is a k-rainbow dominating function of G with weight $\gamma_{rk}(G)$. Note that $\gamma_{r1}(G)$ is the classical domination number $\gamma(G)$. The k-rainbow domination number was introduced by Brešar, Henning, and Rall [1] and has been studied by several authors (see for example [2, 3, 4, 5, 8, 9, 11, 12]).

Following the ideas in [6], we propose a similar game based on the rainbow domination number. Two players $\mathcal{A}$ and $\mathcal{D}$ alternately play on a given graph G, $\mathcal{D}$ playing first, by marking or subdividing an edge of G. An edge which is neither marked nor subdivided is said to be *free*. At the beginning of the game, all the edges of G are free. At each turn, $\mathcal{D}$ marks a free edge of G and $\mathcal{A}$ subdivides a free edge of G by a new vertex. The game ends when all the edges of G are marked or subdivided and results in a new graph G'. The purpose of $\mathcal{D}$ is to minimize the 2-rainbow domination number $\gamma_{r2}(G')$ of G' while $\mathcal{A}$ tries to maximize it. If both $\mathcal{A}$ and $\mathcal{D}$ play according to their optimal strategies, $\gamma_{r2}(G')$ is well defined. We call this number the *rainbow game domination subdivision number* of G and denote it by $\gamma_{rg}(G)$. As the 2-rainbow domination number of any graph obtained by subdividing some of its edges is at least as large as the 2-rainbow domination number of the graph itself, we clearly have $\gamma_{r2}(G) \leq \gamma_{rg}(G)$.

The purpose of this paper is to initiate the study of the rainbow game domination subdivision number of a graph. We first determine $\gamma_{rg}(G)$ for some classes of graphs, and then we establish some bounds on it when G is a tree.

We make use of the following results in this paper.

Proposition A. ([2]) For $n \geq 2$, $\gamma_{r2}(P_n) = \lceil \frac{n+1}{2} \rceil$.

Proposition B. ([2]) For $n \geq 3$, $\gamma_{r2}(C_n) = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{4} \rceil - \lfloor \frac{n}{4} \rfloor$.

The following lower bound for the 2-rainbow domination number of any graph is proved in [8].

Proposition C. For any graph G of order n and maximum degree $\Delta \geq 1$,

$$\gamma_{r2}(G) \geq \frac{2n}{\Delta + 2}.$$

Corollary 1. Let G be an r-regular graph of order n with $r \geq 2$. Then

$$\gamma_{rg}(G) \geq \left\lceil \frac{2(n + \lfloor (rn)/4 \rfloor)}{r+2} \right\rceil.$$

Proof. The graph G has $(rn)/2$ edges. Therefore player $\mathcal{A}$ subdivides $\lfloor (rn)/4 \rfloor$ edges. It follows that the resulting graph G' has maximum degree r and $n + \lfloor (rn)/4 \rfloor$ vertices. Using Proposition C, we deduce that

$$\gamma_{rg}(G) = \gamma_{r2}(G') \geq \left\lceil \frac{2(n + \lfloor (rn)/4 \rfloor)}{r+2} \right\rceil.$$

□

2 Exact value for some classes of graphs

In this section we determined the exact value of the rainbow game domination subdivision number for some classes of graphs.

Example 1. For $n \geq 2$, $\gamma_{rg}(K_{1,n-1}) = \lceil \frac{n+2}{2} \rceil$.

Proof. Clearly $\mathcal{A}$ subdivides exactly $\lfloor \frac{n-1}{2} \rfloor$ edges of $K_{1,n-1}$ and hence $\gamma_{rg}(K_{1,n-1}) = \lfloor \frac{n-1}{2} \rfloor + 2 = \lceil \frac{n+2}{2} \rceil$. □

The *subdivision graph* $S(G)$ is the graph obtained from G by subdividing each edge of G. The subdivision star $S(K_{1,t})$ for $t \geq 2$, is called a *healthy spider* S_t.

Example 2. For every integer $t \geq 2$, $\gamma_{rg}(S(K_{1,t})) = 2t$.

Proof. Let v be the central vertex of $S(K_{1,t})$ and let $N(v) = \{v_1, \ldots, v_t\}$. Assume u_i is the leaf adjacent to v_i. The strategy of $\mathcal{A}$ is as follows. When $\mathcal{D}$ marks an edge in $\{v_iu_i, vv_i\}$, then $\mathcal{A}$ subdivides the other edge in $\{v_iu_i, vv_i\}$, for each $1 \leq i \leq t$. It follows that $\gamma_{rg}(S(K_{1,t})) \geq 2t$. On the other hand, since player $\mathcal{D}$ began the game, he can marks an edge in $\{v_iu_i, vv_i\}$ for each i. Hence $\gamma_{rg}(S(K_{1,t})) = 2t$. □

Example 3. For $n \geq 2$, $\gamma_{rg}(P_n) = \lceil \frac{n}{2} \rceil + \lceil \frac{n-1}{4} \rceil$.

Proof. In the game on a path, all the strategies of $\mathcal{D}$ and $\mathcal{A}$ are equivalent since subdividing any edge of a path results a new path with one more vertex. If $G = P_n$, then $\mathcal{A}$ subdivides $\lfloor \frac{n-1}{2} \rfloor$ edges and $G' = P_{n'}$ with $n' = n + \lfloor \frac{n-1}{2} \rfloor$. Applying Proposition A, we have $\gamma_{r2}(P_{n'}) = \lceil \frac{n'+1}{2} \rceil$ and therefore

$$\gamma_{rg}(P_n) = \left\lceil \frac{n + \lfloor \frac{n-1}{2} \rfloor + 1}{2} \right\rceil = \left\lceil \frac{n}{2} \right\rceil + \left\lceil \frac{n-1}{4} \right\rceil.$$

□

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Using Proposition B and an argument similar to that described in the proof of Example 3 we obtain the next result.

Example 4. For $n \geq 3$, $\gamma_{rg}(C_n) = \lfloor \frac{3n}{4} \rfloor + \lceil \frac{3n-1}{8} \rceil - \lfloor \frac{3n}{8} \rfloor$.

If C_n is a cycle of order $n = 8k$, then Proposition 4 shows that

$$\gamma_{rg}(C_n) = \frac{3n}{4} = \left\lceil \frac{2(n + \lfloor (2n)/4 \rfloor)}{4} \right\rceil.$$

Therefore Corollary 1 is sharp, at least for $r = 2$.

The Dutch-windmill graph, $K_3^{(m)}$, is a graph which consists of m copies of K_3 with a vertex in common.

Example 5. For every positive integer m, $\gamma_{rg}(K_3^{(m)}) = 1 + \lceil \frac{m}{2} \rceil + 2\lfloor \frac{m}{2} \rfloor$.

Proof. Clearly $\gamma_{rg}(K_3) = 2$ and so we assume that $m \geq 2$. Let v, u_i, w_i are the vertices of the i-th copy of K_3 in $K_3^{(m)}$ (v is the common vertex). In the graph $K_3^{(m)'}$ obtained at the end of the game, let p and q be the numbers of cycles whose at most one edge respectively, exactly two edges are subdivided. Then clearly $\gamma_{r2}(K_3^{(m)'}) = 1 + p + 2q$.

The strategy of $\mathcal{D}$ is as follows. When some edge remains free after $\mathcal{A}$ has plaid, $\mathcal{D}$ marks a free edge in a cycle whose two edges are subdivided if possible, otherwise a free edge of cycle that all its edges are free if possible, otherwise a free edge in the cycle whose one edge is marked and one edge is subdivided if possible, otherwise a free edge in the cycle still having free edges. On this way, the number of cycles with exactly two subdivided edges is $\lfloor \frac{m}{2} \rfloor$ and the number of cycles with at most one subdivided edge is $\lceil \frac{m}{2} \rceil$ and hence $\gamma_{rg}(K_3^{(m)}) = \gamma_{r2}(K_3^{(m)'}) \leq 1 + \lceil \frac{m}{2} \rceil + 2\lfloor \frac{m}{2} \rfloor$.

The strategy of $\mathcal{A}$ is as follows. When some edge remains free after $\mathcal{D}$ has plaid, $\mathcal{A}$ subdivides a free edge in a cycle whose one edge is marked and one edge is subdivided if possible, otherwise a free edge in a cycle with two marked edges if possible, otherwise a free edge of cycle that all its edges are free if possible, otherwise a free edge in the cycle still having free edges. On this way, the number of cycles with at least two subdivided edges is $\lfloor \frac{m}{2} \rfloor$ and the number of cycles with one subdivided edge is $\lceil \frac{m}{2} \rceil$. Hence $\gamma_{rg}(K_3^{(m)}) = \gamma_{r2}(K_3^{(m)'}) \geq 1 + \lceil \frac{m}{2} \rceil + 2\lfloor \frac{m}{2} \rfloor$ and the proof is complete. □

For two positive integers p and q, we call a *double star* $DS_{p,q}$ the graph obtained from two stars $K_{1,p}$ of center u and $K_{1,q}$ of center v by adding the edge uv.

Example 6. *For the double star $DS_{1,q}$ of order $n = q + 3 \geq 5$,*

$$\gamma_{rg}(DS_{1,q}) = 2 + \left\lfloor \frac{n+1}{2} \right\rfloor.$$

Proof. By assumption, $q \geq 2$. Then, Player $\mathcal{D}$ cannot prevent $\mathcal{A}$ to subdivide some edge of the star $K_{1,q}$. If $q = 2$, then clearly $\gamma_{rg}(DS_{1,2}) = 5 = 2 + \lfloor \frac{n+1}{2} \rfloor$. Assume henceforth $q \geq 3$. Player $\mathcal{A}$ subdivides $\lfloor \frac{q+2}{2} \rfloor$ edges that among them q' are edges of the star $K_{1,q}$ with $0 < q' \leq \lfloor \frac{q+2}{2} \rfloor < q$. Therefor, the resulting graph $DS'_{1,q}$ has

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2-rainbow domination number $q'+3$ if $q' = \lfloor \frac{q+2}{2} \rfloor$ and $q'+4$ when $q' \leq \lfloor \frac{q}{2} \rfloor$. Hence $\mathcal{D}$ tries to mark and $\mathcal{A}$ to subdivide the largest possible number of edges of the star $K_{1,q}$. At the end of the game, as $\mathcal{D}$ began, $\lfloor \frac{q}{2} \rfloor$ edges of the star are subdivided and $\gamma_{r2}(DS'_{1,q}) = \lfloor \frac{q}{2} \rfloor + 4 = \lfloor \frac{n+1}{2} \rfloor + 2$. □

Example 7. *For the double star $DS_{p,q}$ of order $n = p+q+2$ with $2 \leq p \leq q$,*

$$\gamma_{rg}(DS_{p,q}) = \begin{cases} \frac{n+1}{2} + 2 & \text{if } n \text{ is odd} \\ \frac{n}{2} + 3 & \text{if } n \text{ is even} \end{cases} = \left\lceil \frac{n+1}{2} \right\rceil + 2.$$

Proof. Let p' and q' be the numbers of edges which have been subdivided in the stars $K_{1,p}$ and $K_{1,q}$ respectively, in the graph $DS'_{p,q}$ obtained at the end of the game. Moreover, let $\eta = 1$ if uv is subdivided, $\eta = 0$ otherwise. Clearly $p' + q' + \eta = \lfloor \frac{n-1}{2} \rfloor$ and $p' + q' \leq \lfloor \frac{n-1}{2} \rfloor < n - 2 = p + q$. Then

$$\gamma_{r2}(DS'_{p,q}) = p' + q' + 4 = \left\lfloor \frac{n-1}{2} \right\rfloor - \eta + 4.$$

The strategy of $\mathcal{A}$ is as follows. When some edge remains free after $\mathcal{D}$ has plaid, $\mathcal{A}$ subdivides a free edge in a star already containing marked edges if possible, otherwise a free edge of the star still having the maximum number of free edges if possible, otherwise the edge uv. On this way, $\mathcal{A}$ never simultaneously subdivides uv and all the edges of a star. Hence $\gamma_{r2}(DS'_{p,q}) \geq \lfloor \frac{n-1}{2} \rfloor + 4$. Moreover if n is even, then $\mathcal{A}$ does not subdivide uv, $p' < p$, $q' < q$, $p'+q' = \frac{p+q}{2}$, and $\gamma_{r2}(DS'_{p,q}) = p'+q'+4 = \frac{p+q}{2}+4 = \frac{n-2}{2}+4 = \frac{n}{2}+3$. If n is odd, the total number of edges is even and if $\mathcal{D}$ never marks uv, $\mathcal{A}$ is obliged to subdivide it. Hence $\eta = 1$ and $\gamma_{r2}(DS'_{p,q}) = \lfloor \frac{n-1}{2} \rfloor + 3 = \frac{n+1}{2} + 2$. □

3 2-domination number

In this section we present some sharp bounds on the rainbow game domination subdivision number of graph which deal with to 2-domination.

Proposition 2. Let X be an independent set of G such that $V \setminus X$ is a 2-dominating set. Then $\gamma_{rg}(G) \leq 2(n - |X|)$. In particular, if $\delta(G) \geq 2$ then $\gamma_{rg}(G) \leq 2(n - \alpha(G))$.

Proof. Let $X = \{x_1, \ldots, x_{|X|}\}$ and let $x_i x_i^1, x_i x_i^2 \in E(G)$. First Player $\mathcal{D}$ marks an edge in $E(G) - \{x_i x_i^1, x_i x_i^2 \mid 1 \leq i \leq |X|\}$ if any, otherwise any edge, and continues as follows. When $\mathcal{A}$ subdivides an edge in $\{x_i x_i^1, x_i x_i^2\}$ then $\mathcal{D}$ marks the other free edge in $\{x_i x_i^1, x_i x_i^2\}$ if any, otherwise any free edge. Assume that G' is the graph obtained from G at the end of the game. Obviously, the function $f : V(G') \to \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$ defined by $f(u) = \{1,2\}$ for each $u \in V(G) - X$ and $f(u) = \emptyset$ otherwise, is a 2-rainbow dominating function of G' of weight $2(n - |X|)$. Hence $\gamma_{rg}(G) = \gamma_{r2}(G') \leq 2(n - |X|)$. □

The next result is an immediate consequence of Proposition 2.

Corollary 3. If G is a bipartite graph with $\delta(G) \geq 2$, then $\gamma_{rg}(G) \leq n(G)$.

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Proof. If G is a bipartite graph, then $\alpha(G) \geq n(G)/2$. It follows from Proposition 2 that $\gamma_{rg}(G) \leq 2(n(G) - \alpha(G)) \leq n(G)$. □

Proposition 4. If p and q are two integers with $2 \leq p \leq q$, then

$$\left\lceil \frac{2(p+q+\lfloor (pq)/2 \rfloor)}{q+2} \right\rceil \leq \gamma_{rg}(K_{p,q}) \leq 2p.$$

In particular, $\gamma_{rg}(K_{2,q}) = 4$.

Proof. It follows from Proposition 2 that $\gamma_{rg}(K_{p,q}) \leq 2p$.

The graph $G = K_{p,q}$ has pq edges. Therefore player $\mathcal{A}$ subdivides exactly $\lfloor (pq)/2 \rfloor$ edges. Let G' be the graph obtained at the end of game. Then G' has maximum degree q and $p + q + \lfloor (pq)/2 \rfloor$ vertices. Using Proposition C, we deduce that

$$\gamma_{rg}(G) = \gamma_{r2}(G') \geq \left\lceil \frac{2(p+q+\lfloor (pq)/2 \rfloor)}{q+2} \right\rceil.$$

Now it is easy to see that

$$\left\lceil \frac{2(p+q+\lfloor (pq)/2 \rfloor)}{q+2} \right\rceil \geq p+2,$$

when $q > 3$ and when $p = 2$ and $q > 2$. This implies that $\gamma_{rg}(K_{2,q}) = 4$ when $q > 2$, and $\gamma_{rg}(K_{2,2}) = 4$ follows from Example 4. □

Next we present a sharp upper bound on the rainbow game domination subdivision number of trees.

Remark 5. Consider the variant of the game defined by the same rule with the exception that in one turn of the game, $\mathcal{D}$ is allowed to mark two free edges instead of one. For this variant, the rainbow game domination subdivision number γ'_{rg} satisfies $\gamma'_{rg}(G) \leq \gamma_{rg}(G)$.

For a vertex v in a rooted tree T, let $D(v)$ denote the set of descendants of v and $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v.

Theorem 6. For any tree T of order $n \geq 2$ different from P_3,

$$\gamma_{rg}(T) \leq 2\gamma_2(T) - 2.$$

Furthermore, this bound is sharp for healthy spider.

Proof. The proof is by induction on n. The statement is obviously true for $n \leq 3$. For the inductive hypothesis, let $n \geq 4$ and suppose that for every nontrivial tree T, different from P_3, of order less than n the result is true. Let T be a tree of order n. If T is a star $K_{1,n-1}$, then $\gamma_2(T) = n - 1$ and, by Example 1, $\gamma_{rg}(T) = \lceil \frac{n+2}{2} \rceil$. It follows that $\gamma_{rg}(T) < 2\gamma_2(T) - 2$. If $T = P_4$, then clearly $\gamma_{rg}(T) = 3 < 4 = 2\gamma_2(T) - 2$. If T is a double star $DS_{1,q}$ with $n = 3 + q \geq 5$, then $\gamma_2(T) = n - 1$ and, by Example

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6, $\gamma_{rg}(T) = 2 + \lfloor \frac{n+1}{2} \rfloor$. This implies that $\gamma_{rg}(T) < 2\gamma_2(T) - 2$. If T is a double star $DS_{p,q}$ then by Example 7, $\gamma_{rg}(T) < 2\gamma_2(T) - 2$. Thus, we assume that T is not a star or double star. Then $\mathrm{diam}(T) \geq 4$. Assume that $P = v_1 v_2 \ldots v_k$, $k \geq 5$, is a longest path of T. Let $\deg_T(v_{k-1}) = t$ and let D be a minimum 2-dominating set of T not containing v_{k-1}. Obviously, such a minimum 2-dominating set exists. We consider two cases. In each of them, we define a subtree T_1 of order at least two of T and a strategy for $\mathcal{D}$. We denote by T' and T_1' the trees obtained from T and T_1 at the end of the game.

Case 1. $t \geq 3$.
Root T at v_1 and let $v_k, u_1, \ldots, u_{t-2}$ be the leaves adjacent to v_{k-1}. Then the set $D \setminus \{v_k, u_1, \ldots, u_{t-2}\}$ is a 2-dominating set for the tree $T_1 = T - T_{v_{k-1}}$ and hence

$$\gamma_2(T_1) \leq \gamma_2(T) - (t-1). \tag{1}$$

If $T_1 = P_3$, then it is easy to see that $\gamma_{rg}(T) < 2\gamma_2(T) - 2$. Let $T_1 \neq P_3$. Player $\mathcal{D}$ plays the game according to an optimal strategy on T_1 as long as $\mathcal{A}$ subdivides an edge of T_1. If $\mathcal{A}$ subdivides a free edge in $F = \{v_{k-2}v_{k-1}, v_{k-1}v_k, v_{k-1}u_1, \ldots, v_{k-1}u_{t-2}\}$ then $\mathcal{D}$ marks a free edge in F, if any, and otherwise an arbitrary free edge in T_1, if any. It follows from Remark 5 that $\gamma_{r2}(T_1') \leq \gamma_{rg}(T_1)$. We can extend each $\gamma_{r2}(T_1')$-function, f, to a 2-rainbow dominating function of T' by assigning $\{1, 2\}$ to v_{k-1} and assigning $\{1\}$ to each leaf at distance 2 from v_{k-1}. Thus

$$\gamma_{rg}(T) \leq \gamma_{r2}(T') \leq \gamma_{r2}(T_1') + 2 + \lfloor \frac{t-1}{2} \rfloor \leq \gamma_{rg}(T_1) + 2 + \lfloor \frac{t-1}{2} \rfloor.$$

By the induction hypothesis and (1), we have

$$\gamma_{rg}(T) \leq \gamma_{rg}(T_1) + 2 + \lfloor \frac{t-1}{2} \rfloor \leq (2\gamma_2(T_1) - 2) + 2 + \lfloor \frac{t-1}{2} \rfloor < 2\gamma_2(T) - 2.$$

Case 2. $t = 2$.
Since $v_{k-1} \notin D$, $\{v_k, v_{k-2}\} \subseteq D$ and $D \setminus \{v_k\}$ is a 2-dominating set of the tree $T_1 = T - \{v_k, v_{k-1}\}$. Hence $\gamma_2(T_1) \leq \gamma_2(T) - 1$. If $T_1 = P_3$, then $T = P_5$ and it follows from Example 3 that $\gamma_{rg}(T) < 2\gamma_2(T) - 2$. Let $T_1 \neq P_3$.

Player $\mathcal{D}$ plays the game according to an optimal strategy on T_1 as long as $\mathcal{A}$ subdivides an edge of T_1 and when $\mathcal{A}$ subdivides one edge in $\{v_{k-2}v_{k-1}, v_{k-1}v_k\}$ then $\mathcal{D}$ marks the second edge in $\{v_{k-2}v_{k-1}, v_{k-1}v_k\}$. We may assume, without loss of generality, that $\mathcal{A}$ subdivides the edge $v_{k-1}v_k$ by a new vertex z. We can extend each $\gamma_{r2}(T_1')$-function, f, to a 2-rainbow dominating function of T' by assigning $\{1, 2\}$ to z. Hence

$$\gamma_{rg}(T) \leq \gamma_{rg}(T_1) + 2. \tag{2}$$

It follows from the induction hypothesis and (2) that

$$\gamma_{rg}(T) \leq \gamma_{rg}(T_1) + 2 \leq 2\gamma_2(T_1) - 2 + 2 \leq 2\gamma_2(T) - 2.$$

This completes the proof. □

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4 Trees

In this section we present lower and upper bounds on the rainbow game domination subdivision number of a tree.

Theorem 7. For any tree T of order $n \geq 2$,

$$\gamma_{rg}(T) \geq \lceil \frac{n+2}{2} \rceil.$$

Moreover, $\gamma_{rg}(T) = \lceil \frac{n+2}{2} \rceil$ if and only if $T = P_5$ or T is a star.

Proof. The proof is by induction on n. Obviously, the statement is true for $n \leq 3$. Assume the statement is true for all trees of order less than n, where $n \geq 4$. Let T be a tree of order n. If T is a star, then the result follows from Example 1. If T is a double star, then we deduce from Examples 6 and 7 that $\gamma_{rg}(T) > \lceil \frac{n+2}{2} \rceil$. Suppose T is not a star or double star. Then $\mathrm{diam}(T) \geq 4$. Let $P = v_1 v_2 \ldots v_k$ be a diametral path in T and let $d = \deg_T(v_{k-1})$ and $t = \deg_T(v_2)$. Assume that $u_1, u_2, \ldots, u_{d-2}, v_k$ are the leaves adjacent to v_{k-1} if $d \geq 3$ and $u'_1, u'_1, \ldots, u'_{t-2}, v_1$ are the leaves adjacent to v_2 when $t \geq 3$. In what follows, we will consider trees T_1 formed from T by removing a set of vertices. We denote by T' and T'_1 the trees obtained from T and T_1 at the end of the game. We proceed further with a series of claims that we may assume satisfied by the tree.

Claim 1. $d = 2$ or d is odd.

Suppose $d \geq 3$ and d is even. Let $T_1 = T - \{v_{k-1}, v_k, u_1, \ldots, u_{d-2}\}$. Player $\mathcal{A}$ plays the game according to an optimal strategy on T_1 as long as $\mathcal{D}$ marks an edge of T_1. If $\mathcal{D}$ marks a free edge in $F = \{v_{k-2}v_{k-1}, u_1 v_{k-1}, \ldots, u_{d-2}v_{k-1}, v_{k-1}v_k\}$ then $\mathcal{A}$ subdivides a free edge in F. Suppose that T' is the tree obtained at the end of the game. Obviously, $\mathcal{A}$ subdivides $\frac{d}{2}$ edges in F. Let $\{z_1, z_2, \ldots, z_{\frac{d}{2}}\}$ be the subdivision vertices used to subdivide the edges in F. Then $T' - \{v_{k-1}, v_k, u_1, \ldots, u_{d-2}, z_1, z_2, \ldots, z_{\frac{d}{2}}\}$ is the tree T'_1 obtained from T_1 at the end of the game and $\gamma_{rg}(T_1) = \gamma_{r2}(T'_1)$.

We show that $\gamma_{r2}(T') \geq \gamma_{r2}(T'_1) + \frac{d}{2} + 1$. Let f be a $\gamma_{r2}(T')$-function. If $\mathcal{A}$ has subdivided the edge $v_{k-2}v_{k-1}$, then f must assign $\{1,2\}$ to v_{k-1} and $\{1\}$ to $\frac{d}{2}-1$ leaves at distance 2 from v_{k-1} and hence f assigns $\emptyset$ to the subdivision vertex of the edge $v_{k-1}v_{k-2}$. It follows that the restriction of f to T'_1 is a 2-rainbow dominating function on T'_1 implying that $\gamma_{r2}(T') \geq \gamma_{r2}(T'_1) + \frac{d}{2} + 1$. Let $\mathcal{A}$ don't subdivide the edge $v_{k-2}v_{k-1}$. Then $\mathcal{A}$ has subdivided $\frac{d}{2}$ pendant edges incident to v_{k-1}. Then f must assign $\{1,2\}$ to v_{k-1} and $\{1\}$ to $\frac{d}{2}$ leaves at distance 2 from v_{k-1}. Then the function g defined by $g(v_{k-2}) = \{1\}$ and $g(v) = f(v)$ for each $v \in V(T'_1) - \{v_{k-2}\}$ is a 2-rainbow domination function on T'_1 of with $\omega(f_{T'_1})+1$. It follows that $\gamma_{r2}(T') = \omega(g) + \frac{d}{2} + 1 \geq \gamma_{r2}(T'_1) + \frac{d}{2} + 1$. Thus $\gamma_{r2}(T') \geq \gamma_{r2}(T'_1) + \frac{d}{2} + 1$. Then $\gamma_{rg}(T) \geq \gamma_{rg}(T - \{v_{k-1}, v_k, u_1, \ldots, u_{d-2}\}) + \frac{d}{2} + 1$ and it follows from inductive hypothesis that

$$\begin{aligned} \gamma_{rg}(T) &\geq \gamma_{rg}(T - \{v_{k-1}, v_k, u_1, \ldots, u_{d-2}\}) + \tfrac{d}{2} + 1 \\ &\geq \lceil \tfrac{n-d+2}{2} \rceil + \tfrac{d}{2} + 1 \\ &\geq \lceil \tfrac{n+4}{2} \rceil > \lceil \tfrac{n+2}{2} \rceil. \end{aligned}$$

Similarly, we may assume that $t = 2$ or $t \geq 3$ and t is odd.

Claim 2. $d = 2$ or n is odd.

Suppose $d \geq 3$ and n is even. Then by Claim 1, d is odd. An argument similar to that described in Claim 1, shows that $\gamma_{rg}(T) \geq \lceil \frac{n-d+2}{2} \rceil + \frac{d-1}{2} + 1$. Since n is even, $n - d + 2$ is odd and we have

$$\gamma_{rg}(T) \geq \frac{n-d+3}{2} + \frac{d-1}{2} + 1 = \frac{n+2}{2} + 1 = \lceil \frac{n+2}{2} \rceil + 1 > \lceil \frac{n+2}{2} \rceil.$$

Claim 3. $d = 2$.

Let $d \geq 3$. Then by Claims 1 and 2, the integers d and n are odd. Since $t = 2$ or $t \geq 3$ and t is odd, we consider two Cases.

Case 3.1. $t = 2$.

If $\text{diam}(T) = 4$ and $\deg_T(v_3) = 2$, then n is even which is a contradiction. Hence, we may assume $\text{diam}(T) \geq 5$ or $\deg_T(v_3) \geq 3$. Let $T_1 = T - \{v_1, v_{k-1}, v_k, u_1, \ldots, u_{d-2}\}$. The strategy of $\mathcal{A}$ is that he plays the game according to an optimal strategy on T_1 as long as $\mathcal{D}$ marks edges of T_1. When $\mathcal{D}$ marks an edge in $F = \{u_1v_{k-1}, \ldots, u_{d-2}v_{k-1}, v_{k-1}v_k\}$ then $\mathcal{A}$ subdivides a free edge in F and when $\mathcal{D}$ marks an edge in $\{v_1v_2, v_{k-2}v_{k-1},\}$, then $\mathcal{A}$ subdivides the other edge in $\{v_1v_2, v_{k-2}v_{k-1},\}$. Assume that Z is the set of subdivision vertices used to subdivide the edges not in T_1. Suppose that T' is the tree obtained at the end of the game. Then $T' - (Z \cup \{v_1, v_{k-1}, v_k, u_1, \ldots, u_{d-2}\})$ is the tree T'_1 obtained from T_1 at the end of the game and $\gamma_{rg}(T_1) = \gamma_{r2}(T'_1)$. Using an argument similar to that described in Claim 1, we can see that $\gamma_{r2}(T') \geq \gamma_{r2}(T'_1) + \frac{d-1}{2} + 2$. It follows from induction hypothesis that $\gamma_{r2}(T') \geq \lceil \frac{n-d+1}{2} \rceil + \frac{d-1}{2} + 2 > \lceil \frac{n+2}{2} \rceil$.

Case 3.2. $t \geq 3$ is odd.

If $\text{diam}(T) = 4$ and $\deg_T(v_3) = 2$, then it is easy to verify that $\gamma_{rg}(T) = 4 + \frac{d-1}{2} + \frac{t-1}{2} > \lceil \frac{n+2}{2} \rceil$. Let $\text{diam}(T) \geq 5$ or $\deg_T(v_3) \geq 3$ and let $T_1 = T - (\{v_1, v_2, u'_1, \ldots, u'_{t-2}\} \cup \{v_{k-1}, v_k, u_1, \ldots, u_{d-2}\})$. The strategy of $\mathcal{A}$ is that he plays the game according to an optimal strategy on T_1 as long as $\mathcal{D}$ marks edges of T_1. When $\mathcal{D}$ marks an edge in $F_1 = \{u_1v_{k-1}, \ldots, u_{d-2}v_{k-1}, v_{k-1}v_k\}$ then $\mathcal{A}$ subdivides a free edge in F_1, when $\mathcal{D}$ marks an edge in $F_2 = \{u'_1v_2, \ldots, u'_{t-2}v_2, v_2v_1\}$ then $\mathcal{A}$ subdivides a free edge in F_2 and when $\mathcal{D}$ marks an edge in $\{v_3v_2, v_{k-2}v_{k-1},\}$, then $\mathcal{A}$ subdivides the other edge in $\{v_3v_2, v_{k-2}v_{k-1},\}$. Let Z be the set consists of all subdivision vertices used to subdivide the edges not in T_1 and let T' be the tree obtained at the end of the game. Then $T' - (Z \cup \{v_1, v_2, u'_1, \ldots, u'_{t-2}\} \cup \{v_{k-1}, v_k, u_1, \ldots, u_{d-2}\})$ is the tree T'_1 obtained from T_1 at the end of the game and $\gamma_{rg}(T_1) = \gamma_{r2}(T'_1)$. Using an argument similar to that described in Claim 1, one can see that $\gamma_{r2}(T') \geq \gamma_{r2}(T'_1) + \frac{d-1}{2} + \frac{t-1}{2} + 3$. By inductive hypothesis we have $\gamma_{r2}(T') \geq \lceil \frac{n-d-t+2}{2} \rceil + \frac{d-1}{2} + \frac{t-1}{2} + 3 > \lceil \frac{n+2}{2} \rceil$.

Claim 4. $t = 2$.

Let $t \geq 3$. Then t is odd. Using an argument similar to that described in Case 1 of Claim 3, we can see that $\gamma_{r2}(T') > \lceil \frac{n+2}{2} \rceil$.

Claim 5. $\deg_T(v_{k-2}) = 2$.

Let $\deg_T(v_{k-2}) \geq 3$. We consider three Cases.

Case 1. $\deg_T(v_{k-2}) \geq 3$ and v_{k-2} is adjacent to a support vertex $z_2 \notin \{v_{k-3}, v_{k-1}\}$. By Claims 1, 2, and 3, we may assume $\deg_T(z_2) = 2$. Let z_1 be the leaf adjacent

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to z_2 and let $T_1 = T - \{v_{k-1}, v_k, z_1, z_2\}$. Player $\mathcal{A}$ plays according to an optimal strategy on T_1 as long as $\mathcal{D}$ marks edges of T_1, and when $\mathcal{D}$ marks an edge in $\{v_{k-2}v_{k-1}, v_{k-1}v_k\}$ then $\mathcal{A}$ subdivides the other edge in $\{v_{k-2}v_{k-1}, v_{k-1}v_k\}$ with vertex w_1 and when $\mathcal{D}$ marks an edge in $\{v_{k-2}z_2, z_2z_1\}$ then $\mathcal{A}$ subdivides the other edge in $\{v_{k-2}z_2, z_2z_1\}$ with vertex w_2. Let T' be the tree obtained at the end of the game. Then $T' - \{z_1, z_2, w_1, w_2, v_{k-1}, v_k\}$ is the tree T'_1 obtained from T_1 at the end of game and $\gamma_{rg}(T_1) = \gamma_{r2}(T'_1)$.

Let f be a $\gamma_{r2}(T')$-function. Clearly $|f(w_1)| + |f(v_{k-1})| + |f(v_k)| = 2$, $|f(w_2)| + |f(z_2)|+|f(z_1)| = 2$, and the function $g : V(T'_1) \to \mathcal{P}(\{1, 2\})$ defined by $g(v_{k-2}) = \{1\}$ and $g(x) = f(x)$ for each $x \in V(T'_1) - \{v_{k-2}\}$, is a 2RDF of T'_1 of weight $\omega(f) - 3$. Hence $\gamma_{r2}(T') = \omega(f) = \omega(g) + 3 \geq \gamma_{r2}(T'_1) + 3 \geq \lceil\frac{n-4+2}{2}\rceil + 3 > \lceil\frac{n+2}{2}\rceil$.

Case 2. $\deg_T(v_{k-2}) \geq 3$ and v_{k-2} is adjacent to two leaves z_1, z_2.
Let $T_1 = T - \{v_{k-1}, v_k, z_1, z_2\}$. Player $\mathcal{A}$ plays according to an optimal strategy on T_1 as long as $\mathcal{D}$ marks edges of T_1, and when $\mathcal{D}$ marks an edge in $\{v_{k-2}v_{k-1}, v_{k-1}v_k\}$ then $\mathcal{A}$ subdivides the other edge in $\{v_{k-2}v_{k-1}, v_{k-1}v_k\}$ and when $\mathcal{D}$ marks an edge in $\{v_{k-2}z_2, v_{k-2}z_1\}$ then $\mathcal{A}$ subdivides the other edge in $\{v_{k-2}z_2, v_{k-2}z_1\}$. Let T' be the tree obtained at the end of the game. As above, one can see that $\gamma_{r2}(T') \geq \lceil\frac{n-4+2}{2}\rceil + 3 > \lceil\frac{n+2}{2}\rceil$.

Case 3. $\deg_T(v_{k-2}) = 3$ and v_{k-2} is adjacent to the leaf z_1.
Let $T_1 = T - \{v_{k-2}, v_{k-1}, v_k, z_1\}$. Player $\mathcal{A}$ plays according to an optimal strategy on T_1 as long as $\mathcal{D}$ marks an edge of T_1, and when $\mathcal{D}$ marks an edge in $\{v_{k-2}v_{k-1}, v_{k-1}v_k\}$ then $\mathcal{A}$ subdivides the other edge in $\{v_{k-2}v_{k-1}, v_{k-1}v_k\}$ and when $\mathcal{D}$ marks an edge in $\{v_{k-2}v_{k-3}, v_{k-2}z_1\}$ then $\mathcal{A}$ subdivides the other edge in $\{v_{k-2}v_{k-3}, v_{k-2}z_1\}$. If T' is the tree obtained at the end of the game then as above, we can see that $\gamma_{r2}(T') \geq \lceil\frac{n-4+2}{2}\rceil + 3 > \lceil\frac{n+2}{2}\rceil$.

Similarly, we may assume $\deg(v_3) = 2$.

We now return to the proof of theorem. If $\mathrm{diam}(T) = 4$, then $T = P_5$ and clearly $\gamma_{rg}(T) = 4 = \lceil\frac{n+2}{2}\rceil$. If $\mathrm{diam}(T) = 5$ or $\mathrm{diam}(T) = 6$ and $\deg(v_4) = 1$, then $T = P_6, P_7$ and $\gamma_{rg}(T) > \lceil\frac{n+2}{2}\rceil$ by Example 3. Let $\mathrm{diam}(T) > 6$ or $\deg(v_4) \geq 3$. Suppose $T_1 = T - \{v_1, v_2, v_3, v_{k-2}, v_{k-1}, v_k\}$. Player $\mathcal{A}$ plays according to an optimal strategy on T_1 as long as $\mathcal{D}$ marks edges of T_1, and when $\mathcal{D}$ marks an edge in $\{v_{k-2}v_{k-1}, v_{k-1}v_k\}$ then $\mathcal{A}$ subdivides the other edge in $\{v_{k-2}v_{k-1}, v_{k-1}v_k\}$, when $\mathcal{D}$ marks an edge in $\{v_1v_2, v_2v_3\}$ then $\mathcal{A}$ subdivides the other edge in $\{v_1v_2, v_2v_3\}$ and when $\mathcal{D}$ marks an edge in $\{v_{k-2}v_{k-3}, v_3v_4\}$ then $\mathcal{A}$ subdivides the other edge in $\{v_{k-2}v_{k-3}, v_3v_4\}$. If T' is the tree obtained at the end of the game, then it is not hard to see that $\gamma_{r2}(T') \geq \lceil\frac{n-4+2}{2}\rceil + 3 > \lceil\frac{n+2}{2}\rceil$.

All in all, we have $\gamma_{r2}(T') \geq \lceil\frac{n+2}{2}\rceil$ with equality if and only if $T = P_5$ or T is a star. This completes the proof. □

A support vertex is said to be *end-support vertex* if all its neighbors except one of them are leaves.

Theorem 8. For any tree T of order $n \geq 2$,

$$\gamma_{rg}(T) \leq n.$$

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Proof. The proof is by induction on n. If $n = 2, 3$, then obviously, $\gamma_{rg}(T) = n$. Let $n \geq 4$. Assume that the result is true for any non-trivial tree of order less than n, and let T be a tree of order n. If T is a star, then $\gamma_{rg}(T) < n$ by Example 1 and $n \geq 4$. If T is a double star, then it follows from Examples 6 and 7 that $\gamma_{rg}(T) \leq n$ with equality if and only if $T = DS_{1,2}$ or $DS_{2,2}$. Assume that T is not a star or a double star. Then $\text{diam}(T) \geq 4$. Let x be an end-support vertex of degree $\deg_T(x) = t$ of T, $y^1, y^2, \cdots, y^{t-1}$ the leaves attached at x, and z the neighbor of x of degree at least 2. The tree $T_1 = T - \{x, y^1, y^2, \cdots, y^{t-1}\}$ has order at least two. In the following three cases, we define a strategy for $\mathcal{D}$ and denote by T' and T_1' the trees obtained from T and T_1 at the end of the game.

Case 1. The tree T has an end-support vertex of degree at least 5.
Player $\mathcal{D}$ plays following its best strategy on T_1 as long as $\mathcal{A}$ subdivides edges of T_1. When $\mathcal{A}$ subdivides an edge of $\{xz, xy^1, \ldots, xy^{t-1}\}$, then $\mathcal{D}$ marks a free edge in $\{xy^1, \ldots, xy^{t-1}\}$. It is easy to see that $\gamma_{r2}(T') \leq \gamma_{r2}(T_1') + \lfloor \frac{t}{2} \rfloor + 2$. Hence, by the induction hypothesis and $t \geq 5$,

$$\gamma_{rg}(T) = \gamma_{r2}(T') \leq \gamma_{r2}(T_1') + \lfloor \frac{t}{2} \rfloor + 2 = \gamma_{rg}(T_1') + \lfloor \frac{t}{2} \rfloor + 2 \leq n - t + \lfloor \frac{t}{2} \rfloor + 2 < n.$$

Case 2. T admits two end-support vertices x, x' of degree 4.
Assume that y'^1, y'^2, y'^3 are the leaves attached at x', and z' the neighbor of x' of degree at least 2. Suppose that $T_2 = T - \{x, y^1, y^2, y^3, x', y'^1, y'^2, y'^3\}$. If $T_2 = K_1$, then $z = z'$ and it is easy to see that $\gamma_{rg}(T) < n$. Let T_2 have order at least two. The strategy of $\mathcal{D}$ is that he plays its best strategy on T_2 as long as $\mathcal{A}$ subdivides edges of T_2. When $\mathcal{A}$ subdivides an edge of $\{xy^1, xy^2, xy^3, x'y'^1, x'y'^2, x'y'^3\}$, $\mathcal{D}$ marks a free edge of $\{xy^1, xy^2, xy^3, x'y'^1, x'y'^2, x'y'^3\}$ and when $\mathcal{A}$ subdivides an edge in $\{xz, x'z'\}$, $\mathcal{D}$ marks the other edge in $\{xz, x'z'\}$. Clearly $\gamma_{r2}(T') \leq \gamma_{r2}(T_2') + 7$. By the inductive hypothesis, we have

$$\gamma_{rg}(T) = \gamma_{r2}(T') \leq \gamma_{r2}(T_2') + 7 = \gamma_{rg}(T_2) + 7 \leq n - 8 + 7 < n.$$

Case 3. All the end-support vertices of T have degree at most 4 and at most one of them has degree 4.
Let x be an end-support vertex of degree t. Player $\mathcal{D}$ plays following its best strategy on T_1 as long as $\mathcal{A}$ subdivides edges of T_1. When $\mathcal{A}$ subdivides an edge of $\{xy^1, xy^2, \cdots, xy^{t-1}, xz\}$, $\mathcal{D}$ marks an edge of $\{xy^1, xy^2, \cdots, xy^{t-1}\}$ if possible, otherwise the edge xz if still free, otherwise any other free edge of T_1. At the end of the game, at most $\lfloor \frac{t}{2} \rfloor$ edges of $xy^1, xy^2, \cdots, xy^{t-1}$ are subdivided. It is easy to see that $\gamma_{r2}(T') \leq \gamma_{r2}(T_1') + 2$ when $t = 2$ and $\gamma_{r2}(T') \leq \gamma_{r2}(T_1') + 2 + \lfloor \frac{t}{2} \rfloor$ when $t = 3, 4$. It follows from induction hypothesis that $\gamma_{rg}(T) \leq \gamma_{r2}(T') \leq n$ and the proof is complete. □

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