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Stability of alternate dual frames

Ali Akbar Arefijamaal

Abstract. The stability of frames under perturbations, which is important in applications, is studied by many authors. It is worthwhile to consider alternate duals instead of canonical duals in the frame decomposition. In this paper, we apply a new characterization of alternate dual frames to discuss a stability problem for them.

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1. Introduction and preliminaries

Let $\mathcal{H}$ be a separable Hilbert space. A sequence $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is called a *frame* for $\mathcal{H}$ if there are constants $A, B > 0$ satisfying

$$A\|f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, \qquad (f \in \mathcal{H}). \tag{1.1}$$

If the right-hand side of (1.1) holds, it is said to be a Bessel sequence. If $\{f_i\}_{i=1}^{\infty}$ is a frame, the *frame operator* is defined by

$$S : \mathcal{H} \to \mathcal{H}, \qquad Sf = \sum_{i=1}^{\infty} \langle f, f_i\rangle f_i.$$

This series converging unconditionally and S is a bounded, invertible, and self-adjoint operator. This leads to the frame decomposition

$$f = S^{-1}Sf = \sum_{i=1}^{\infty} \langle f, S^{-1}f_i\rangle f_i, \qquad (f \in \mathcal{H}). \tag{1.2}$$

A sequence $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is called a *dual frame* for $\{f_i\}_{i=1}^{\infty}$ if

$$f = \sum_{i=1}^{\infty} \langle f, g_i\rangle f_i, \tag{1.3}$$

Every frame at least has a dual. In fact, if $\{f_i\}_{i=1}^{\infty}$ is a frame, then (1.2) follows that $\{S^{-1}f_i\}_{i=1}^{\infty}$, which is a frame with bounds B^{-1} and A^{-1}, is a dual for $\{f_i\}_{i=1}^{\infty}$; it is called the *canonical dual*, a dual which is not the canonical dual is called an *alternate dual*, or simply a dual. To see a general text in frames theory see [8].

Recently, various generalization of frames have been proposed. For example, continuous frames [1, 2, 13], g-frames [17], fusion frames [7], von Neumann-Schatten frames [16] and so on. Due to the wide variety of applications, there has been a great influx of researchers into the subject in different approaches, cf. [3, 4, 5, 14].

Let $\{f_i\}_{i=1}^{\infty}$ be a frame in $\mathcal{H}$ and $\{g_i\}_{i=1}^{\infty}$ in some sense be closed to $\{f_i\}_{i=1}^{\infty}$, does it follow that $\{g_i\}_{i=1}^{\infty}$ is also a frame? This question, which plays an essential role in applications, is referred as the stability of frames. The stability of frames was introduced by Favier and Zalik [11]. Although, it is developed by many other authors [6, 15] in the last decade, but most of known results on this topic, are stated about canonical dual, see [12] for frames and [17, 18] for g-frames. Not only computing the canonical dual is highly nontrivial but also the structure of a frame is not shared by its canonical dual, in general [10]. Hence, we try to choose a suitable dual instead of a canonical dual. In this article, we first review some characterizations of dual frames. Then we discuss a perturbation property for alternate duals.

2. Main Results

Let $\{f_i\}_{i=1}^{\infty}$ be a frame in $\mathcal{H}$ with the frame operator S. For each $f \in \mathcal{H}$ due to Lemma 5.3.6 of [8] the sequence $\{\langle f, S^{-1}f_i\rangle\}_{i=1}^{\infty}$ has minimal l^2-norm among all sequences $\{\langle f, g_i\rangle\}_{i=1}^{\infty}$ where $\{g_i\}_{i=1}^{\infty}$ is an arbitrary alternate dual. However, finding alternate duals is complicated and disorderly, see [8, 9]. In order to derive our characterizations of duals we must define the synthesis operator. If $\{f_i\}_{i=1}^{\infty}$ is a Bessel sequence for $\mathcal{H}$, then the operator $T : l^2 \to \mathcal{H}$ defined by

$$T\{c_i\}_{i=1}^{\infty} := \sum_{i=1}^{\infty} c_i f_i.$$

is a bounded linear operator, called *the synthesis operator* or the *pre-frame operator*. It is surjective and bounded if and only if $\{f_i\}_{i=1}^{\infty}$ is a frame. Moreover, $S = TT^*$, where $T^* : \mathcal{H} \to l^2; f \mapsto \{\langle f, f_i\rangle\}$ is the adjoint of T [8].

The next lemma follows immediately from the definition.

Lemma 2.1. *Let $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ be two frames with the synthesis operators T and U, respectively. Then the following are equivalent:*

1. $\{f_i\}_{i=1}^{\infty}$ *is a dual for* $\{g_i\}_{i=1}^{\infty}$.
2. $TU^* = I$.
3. $UT^* = I$.
4. $(T^*U)^2 = T^*U$.

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The following theorem describe the alternate dual in terms of the left inverse of the synthesis operator.

Theorem 2.2. [8] *Let $\{f_i\}_{i=1}^{\infty}$ be a frame for $\mathcal{H}$. The dual frames for $\{f_i\}_{i=1}^{\infty}$ are precisely as $\{\Phi\delta_i\}_{i=1}^{\infty}$, where $\Phi : l^2 \to \mathcal{H}$ is a bounded left inverse of T^*.*

The following characterization of alternate duals shows that the difference between an alternate dual and the canonical dual can be considered as a bounded operator.

Theorem 2.3. *Let $\{f_i\}_{i=1}^{\infty}$ be a frame for $\mathcal{H}$ with the bounds A and B and the synthesis operator T. Then there exists an one to one correspondence between duals of $\{f_i\}_{i=1}^{\infty}$ and operators $\Psi \in B(\mathcal{H}, l^2)$ such that $T\Psi = 0$.*

Proof. First assume that $\{g_i\}_{i=1}^{\infty}$ is a dual of $\{f_i\}_{i=1}^{\infty}$. Denote the synthesis operator of $\{g_i\}_{i=1}^{\infty}$ by U. Define $\Psi : \mathcal{H} \to l^2$ by

$$\Psi f = U^* f - T^*(TT^*)^{-1} f.$$

Clearly, Ψ is a bounded operator and by using lemma 2.1 we have

$$T\Psi f = TU^* f - TT^*(TT^*)^{-1} f = 0.$$

Conversely, let $\Psi \in B(\mathcal{H}, l^2)$ with $T\Psi = 0$, also let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis of l^2. Take

$$g_i = S^{-1} f_i + \Psi^*(e_i),$$

where S is the frame operator of $\{f_i\}_{i=1}^{\infty}$. Then

$$\begin{aligned}\sum_{i=1}^{\infty} |\langle f, g_i\rangle|^2 &\leq \sum_{i=1}^{\infty} |\langle f, S^{-1} f_i\rangle + \langle f, \Psi^* e_i\rangle|^2 \\ &\leq \left(A^{-1} + \|\Psi\|^2 + 2\sqrt{A^{-1}}\|\Psi\|\right)\|f\|^2,\end{aligned}$$

for all $f \in \mathcal{H}$. This follows that $\{g_i\}_{i=1}^{\infty}$ is a Bessel sequence.

Moreover,

$$UT^* = \sum_{i=1}^{\infty} \langle f, f_i\rangle g_i = \sum_{i=1}^{\infty} \langle f, f_i\rangle S^{-1} f_i + \langle f, f_i\rangle \Psi^*(e_i) = f + \Psi^* T^* f = f.$$

Therefore, $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ are dual of each other by Lemma 2.1. It is not difficult to see that every two Bessel sequences $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ satisfying in (1.3) must be frames. □

Now we are ready to state our main result about the stability of alternate duals.

Theorem 2.4. *Let $\{f_i\}_{i=1}^{\infty}$ and $\{f'_i\}_{i=1}^{\infty}$ be two frames for $\mathcal{H}$ with the bounds A_1, B_1 and A_2, B_2, respectively. Also let $\{g_i\}_{i=1}^{\infty}$ be a fix alternate dual for $\{f_i\}_{i=1}^{\infty}$. If $\{f_i - f'_i\}_{i=1}^{\infty}$ is a Bessel sequence with sufficiently small bound $\epsilon > 0$, then there exists an alternate dual $\{g'_i\}_{i=1}^{\infty}$ for $\{f'_i\}_{i=1}^{\infty}$ such that $\{g_i - g'_i\}_{i=1}^{\infty}$ is also Bessel and its bound is a multiple of ϵ.*

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Proof. Assume that T_1 and T_2 are the synthesis operators of, $\{f_i\}_{i=1}^{\infty}$ and $\{f_i'\}_{i=1}^{\infty}$, respectively. Also denote $S_1 = T_1T_1^*$ and $S_2 = T_2T_2^*$ as their frame operators. Due to the proof of Theorem 2.3 there exits a $\Psi \in B(\mathcal{H}, l^2)$ such that $T_1\Psi = 0$ and

$$g_i = S_1^{-1}f_i + \Psi^*(e_i), \tag{2.1}$$

where $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis of $\mathcal{H}$. Take

$$h_i = S_2^{-1}f_i' + \Psi^*(e_i). \tag{2.2}$$

Obviously that $\{h_i\}_{i=1}^{\infty}$ is a Bessel sequence with the bound $A_2^{-1} + \|\Psi\|^2 + 2A_2^{\frac{-1}{2}}\|\Psi\|$. Denote the synthesis operator of $\{h_i\}_{i=1}^{\infty}$ by T_3, then

$$\begin{aligned} \|f - T_2T_3^*f\| &= \left\|f - \sum_i \langle f, h_i\rangle f_i'\right\| \\ &= \|T_2\Psi f\| \\ &= \|T_2\Psi f - T_1\Psi f\| \\ &\leq \|T_1 - T_2\|\|\Psi\|\|f\| \leq \epsilon\|\Psi\|\|f\|, \end{aligned}$$

Therefore, $T_2T_3^*$ is invertible for sufficiently small $\epsilon > 0$. In particular,

$$\|I - T_2T_3^*\| \leq \epsilon\|\Psi\|. \tag{2.3}$$

Put $\{g_i'\}_{i=1}^{\infty} = \{(T_2T_3^*)^{-1}h_i\}_{i=1}^{\infty}$. We are going to show that is the desired frame.

On the other hand,

$$\begin{aligned} \|S_1 - S_2\| &= \|T_1T_1^* - T_1T_2^* + T_1T_2^* - T_2T_2^*\| \\ &= \|T_1 - T_2\|(\|T_1\| + \|T_2\|) \\ &\leq \epsilon(\sqrt{B_1} + \sqrt{B_2}). \end{aligned}$$

Moreover, by using (2.1) and (2.2) for all finite scalars c_i we obtain

$$\begin{aligned} \left\|\sum_{i=1}^{\infty} c_i(g_i - h_i)\right\| &= \left\|\sum_{i=1}^{\infty} c_i(S_1^{-1}f_i - S_2^{-1}f_i')\right\| \\ &\leq \left\|\sum_{i=1}^{\infty} c_i(S_1^{-1}f_i - S_2^{-1}f_i)\right\| + \left\|\sum_{i=1}^{\infty} c_i(S_2^{-1}f_i - S_2^{-1}f_i')\right\| \\ &\leq \|S_1^{-1} - S_2^{-1}\|\,\|T_1c_i\| + \|S_2^{-1}\|\,\|T_1c_i - T_2c_i\| \\ &\leq (\|S_2^{-1}\|\|S_2 - S_1\|\|S_1^{-1}\|\|T_1\| + \|S_2^{-1}\|\|T_1 - T_2\|)\left(\sum_{i=1}^{\infty}|c_i|^2\right)^{\frac{1}{2}} \\ &\leq \frac{\epsilon}{A_2}\left(\frac{B_1 + \sqrt{B_1B_2}}{A_1} + 1\right)\left(\sum_{i=1}^{\infty}|c_i|^2\right)^{\frac{1}{2}}. \end{aligned}$$

If take $W = (T_3T_2^*)^{-1}$, then we obtain from (2.3),

$$\|W\| \leq \frac{1}{1 - \|I - W^{-1}\|} < \frac{1}{1 - \epsilon\|\Psi\|},$$

and

$$\|I-W\| = \|W\|\,\|I-W^{-1}\| < \frac{\epsilon\|\Psi\|}{1-\epsilon\|\Psi\|}.$$

Consequently,

$$\begin{aligned}\left\|\sum_{i=1}^{\infty} c_i(g_i-g_i')\right\| &= \left\|\sum_{i=1}^{\infty} c_i(g_i-Wg_i+Wg_i-Wh_i)\right\| \\ &\le \|I-W\|\left\|\sum_{i=1}^{\infty} c_i g_i\right\| + \|W\|\left\|\sum_{i=1}^{\infty} c_i(g_i-h_i)\right\| \\ &\le \epsilon\left(\|\Psi\|\sqrt{B'}+\frac{B_1+\sqrt{B_1B_2}}{A_1A_2}+\frac{1}{A_2}\right)\frac{\left(\sum_{i=1}^{\infty}|c_i|^2\right)^{\frac{1}{2}}}{1-\epsilon\|\Psi\|}\end{aligned}$$

where B' is the upper bound of $\{g_i\}_{i=1}^{\infty}$. This completes the proof. □

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Ali Akbar Arefijamaal
Department of Mathematics and Computer Sciences, Hakim Sabzevari University
Sabzevar, Iran.

e-mail: `Arefijamaal@hsu.ac.ir, Arefijamaal@gmail.com`

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