

# SEMI-SYMMETRIC NON-METRIC CONNECTIONS ON KENMOTSU MANIFOLDS

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ABSTRACT. The object of the present paper is to study a type of semi-symmetric non-metric connection on a Kenmotsu manifold.

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## 1. INTRODUCTION

Manifolds known as Kenmotsu manifolds have been studied by K. Kenmotsu in 1972 [14]. They set up one of the three classes of almost contact Riemannian manifolds whose automorphism group attains the maximum dimension [34]. Consider an almost contact metric manifold  $M^{2n+1}$ , with structure  $(\phi, \xi, \eta, g)$  given by a tensor field  $\phi$  of type  $(1, 1)$ , a vector field  $\xi$ , a 1-form  $\eta$  satisfying  $\phi^2 = -I + \eta \otimes \xi$ ,  $\eta(\xi) = 1$ , and a Riemannian metric  $g$  such that  $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$  for any vector fields  $X$  and  $Y$ . The fundamental 2-form  $\Phi$  is defined by  $\Phi(X, Y) = g(X, \phi Y)$  for any vector fields  $X$  and  $Y$ . The normality of an almost contact metric manifold is expressed by the vanishing of the tensor field  $N = [\phi, \phi] + 2d\eta \otimes \xi$ , where  $[\phi, \phi]$  is the Nijenhuis tensor of  $\phi$  [6]. For more details we refer to Blair's books ([6],[7]). A Kenmotsu manifold can be defined as a normal almost contact metric manifold such that  $d\eta = 0$  and  $d\Phi = 2\eta \wedge \Phi$ . It is well known that Kenmotsu manifolds can be characterized, through their Levi-Civita connection, by  $(\nabla_X \phi)(Y) = g(\phi X, Y)\xi - \eta(Y)\phi X$ , for any vector fields  $X, Y, Z$ . Moreover Kenmotsu proved that such a manifold  $M^{2n+1}$  is locally a warped product  $]-\varepsilon, \varepsilon[_f \times_f N^{2n}$ ,  $N^{2n}$  being a Kähler manifold and  $f^2 = ce^{2t}$ ,  $c$  a positive constant.

More recently in ([15],[18]) and [11], almost contact metric manifolds such that  $\eta$  is closed and  $d\Phi = 2\eta \wedge \Phi$  are studied and they are called *almost Kenmotsu*. Obviously a normal almost Kenmotsu manifold is a Kenmotsu manifold.

In 1924, Friedmann and Schouten [12] introduced the idea of semi-symmetric connection on a differentiable manifold. A linear connection  $\bar{\nabla}$  on a differentiable manifold  $M$  is said to be a *semi-symmetric connection* if the torsion tensor  $T$  of the connection  $\bar{\nabla}$  satisfies

$$(1.1) \quad T(X, Y) = u(Y)X - u(X)Y,$$

where  $u$  is a 1-form and  $\rho$  is a vector field defined by

$$(1.2) \quad u(X) = g(X, \rho),$$

for all vector fields  $X \in \chi(M)$ ,  $\chi(M)$  is the set of all differentiable vector fields on  $M$ .

In 1932, Hayden [13] introduced the idea of semi-symmetric metric connections on a Riemannian manifold  $(M, g)$ . A semi-symmetric connection  $\tilde{\nabla}$  is said to be a *semi-symmetric metric connection* if

$$(1.3) \quad \tilde{\nabla}g = 0.$$

A relation between the semi-symmetric metric connection  $\tilde{\nabla}$  and the Levi-Civita connection  $\nabla$  of  $(M, g)$  was given by Yano [35]:  $\tilde{\nabla}_X Y = \nabla_X Y + u(Y)X - g(X, Y)\rho$ , where  $u(X) = g(X, \rho)$ .

The study of semi-symmetric metric connection was further developed by Amur and Pujar [2], Binh [5], De [8], Singh et al. [28], Ozgur et al ([19],[20]), Ozen, Uysal and Demirbag [22] and many others.

After a long gap the study of a semi-symmetric connection  $\bar{\nabla}$  satisfying

$$(1.4) \quad \bar{\nabla}g \neq 0.$$

was initiated by Prvanović [24] with the name pseudo-metric semi-symmetric connection and was just followed by Andonie [3].

A semi-symmetric connection  $\bar{\nabla}$  is said to be a *semi-symmetric non-metric connection* if it satisfies the condition (1.4).

In 1992, Agashe and Chafle [1] studied a semi-symmetric non-metric connection  $\bar{\nabla}$ , whose torsion tensor  $T$  satisfies  $T(X, Y) = u(Y)X - u(X)Y$  and  $(\bar{\nabla}_X g)(Y, Z) = -u(Y)g(X, Z) - u(Z)g(X, Y)$ . They proved that the projective curvature tensor of the manifold with respect to these two connections are equal to each other. In 1992, Barua and Mukhopadhyay [4] studied a type of semi-symmetric connection  $\bar{\nabla}$  which satisfies  $(\bar{\nabla}_X g)(Y, Z) = 2u(X)g(Y, Z) - u(Y)g(X, Z) - u(Z)g(X, Y)$ . Since  $\bar{\nabla}g \neq 0$ , this is another type of semi-symmetric non-metric connection. However, the authors preferred the name *semi-symmetric semimetric connection*.

In 1994, Liang [17] studied another type of semi-symmetric non-metric connection  $\bar{\nabla}$  for which we have  $(\bar{\nabla}_X g)(Y, Z) = 2u(X)g(Y, Z)$ , where  $u$  is a non-zero 1-form and he called this a *semi-symmetric recurrent metric connection*.

The semi-symmetric non-metric connections was further developed by several authors such as De and Biswas [9], De and Kamilya [10], Liang [17], Singh et al. ([27], [29], [30]), Ozen, Demirbag, Ussal and Yilmaz [21], Smaranda [25], Smaranda and Andonie [26] and many others

A Riemannian manifold is said to be *semi-symmetric* ([31], [32], [33], [16]) if the curvature tensor  $R$  with respect to the Riemannian connection satisfies the condition

$$R(X, Y) \bullet R = 0,$$

where  $R(X, Y)$  denotes the derivation of the tensor algebra at each point of the manifold for tangent vectors  $X, Y$ .

Semi-symmetric manifolds are the generalization of manifolds of constant curvature and locally symmetric manifolds (i.e.,  $\nabla R = 0$ ). Also Ricci semi-symmetric manifolds are the generalizations of semi-symmetric and Einstein manifolds.

In this paper we study a type of semi-symmetric non-metric connection due to Barua and Mukhopadhyay [4] on Kenmotsu manifolds. The paper is organized as follows : After introduction in Section 2, we

give a brief account of the Kenmotsu manifolds. In Section 3, we define a semi-symmetric non-metric connection. Section 4 deals with the relation of the curvature tensor between the Levi-Civita connection and the semi-symmetric non-metric connection on a Kenmotsu manifold and we also construct an example of a 5-dimensional Kenmotsu manifold admitting a type of semi-symmetric non-metric connection whose curvature tensor satisfies the skew-symmetric property and the first Bianchi identity. A Kenmotsu manifold whose curvature tensor is covariant constant with respect to the semi-symmetric non-metric connection and  $M$  is recurrent with respect to the Levi-Civita connection have been studied in Section 5. Finally, we have classified Kenmotsu manifolds which satisfy  $\bar{R} \bullet \bar{R} = 0$  with respect to the semi-symmetric non-metric connection, where  $\bar{R}$  is the curvature tensor with respect to the semi-symmetric non-metric connection.

## 2. KENMOTSU MANIFOLDS

Let  $M$  be an  $(2n + 1)$ -dimensional almost contact metric manifold with an almost contact metric structure  $(\phi, \xi, \eta, g)$  consisting of a  $(1, 1)$  tensor field  $\phi$ , a vector field  $\xi$ , a 1-form  $\eta$  and the Riemannian metric  $g$  on  $M$  satisfying [6]

$$(2.1) \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta(\phi(X)) = 0, \quad g(X, \xi) = \eta(X),$$

$$(2.2) \quad \phi^2(X) = -X + \eta(X)\xi,$$

$$(2.3) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for all vector fields  $X, Y$  on  $\chi(M)$ . A manifold with the almost contact metric structure  $(\phi, \xi, \eta, g)$  is an almost Kenmotsu manifold if the following conditions are satisfied

$$d\eta = 0; \quad d\Omega = 2\eta \wedge \Omega,$$

where  $\Omega$  being the 2-form defined by  $\Omega(X, Y) = g(X, \phi Y)$ . Any normal almost Kenmotsu manifold is a Kenmotsu manifold. An almost contact metric structure  $(\phi, \xi, \eta, g)$  is a Kenmotsu manifold [14] if and only if

$$(2.4) \quad (\nabla_X \phi)(Y) = g(\phi X, Y)\xi - \eta(Y)\phi X.$$

Hereafter we denote the Kenmotsu manifold of dimension  $(2n + 1)$  by  $M$ . From the above relations, it follows that

$$(2.5) \quad \nabla_X \xi = X - \eta(X)\xi,$$

$$(2.6) \quad (\nabla_X \eta)(Y) = g(X, Y) - \eta(X)\eta(Y),$$

$$(2.7) \quad R(X, Y)\xi = \eta(X)Y - \eta(Y)X,$$

$$(2.8) \quad R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi,$$

$$(2.9) \quad \eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X),$$

$$(2.10) \quad S(X, \xi) = -2n\eta(X),$$

where  $R$  and  $S$  denote the curvature tensor and the Ricci tensor of  $M$ , respectively, with respect to the Levi-Civita connection.

Let  $M$  be a Kenmotsu manifold.  $M$  is said to be a  $\eta$ -Einstein manifold if there exists the real valued functions  $\lambda_1, \lambda_2$  such that

$$S(X, Y) = \lambda_1 g(X, Y) + \lambda_2 \eta(X)\eta(Y).$$

For  $\lambda_2 = 0$ , the manifold  $M$  is an Einstein manifold.

Now we state the following:

**Lemma 2.1.** [14] *Let  $M$  be an  $\eta$ -Einstein Kenmotsu manifold of the form  $S(X, Y) = \lambda_1 g(X, Y) + \lambda_2 \eta(X)\eta(Y)$ . If  $\lambda_2 = \text{constant}$  (or,  $\lambda_1 = \text{constant}$ ), then  $M$  is an Einstein one.*

### 3. SEMI-SYMMETRIC NON-METRIC CONNECTION

This section deals with a type of semi-symmetric non-metric connection on a Kenmotsu manifold. We consider a type of semi-symmetric non-metric connection which was introduced by Barua and Mukhopadhyay [4]. In a Kenmotsu manifold a semi-symmetric non-metric connection is defined by  $T(X, Y) = \eta(Y)X - \eta(X)Y$ , with  $\xi$  as the associated vector field, i.e.,  $g(X, \xi) = \eta(X)$ .

A relation between semi-symmetric non-metric connection  $\bar{\nabla}$  and the Levi-Civita connection  $\nabla$  have been obtained by Barua and Mukhopadhyay [4] and is given by

$$(3.1) \quad \bar{\nabla}_X Y = \nabla_X Y - \eta(X)Y + g(X, Y)\xi.$$

Using (3.1), the torsion tensor  $T$  of  $M$  with respect to the connection  $\bar{\nabla}$  is given by

$$(3.2) \quad T(X, Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X, Y] = \eta(Y)X - \eta(X)Y.$$

Hence a relation satisfying (3.2) is called a semi-symmetric connection.

Further using (3.1), we have

$$(3.3) \quad \begin{aligned} (\bar{\nabla}_X g)(Y, Z) &= \nabla_X g(Y, Z) - g(\bar{\nabla}_X Y, Z) - g(Y, \bar{\nabla}_X Z) \\ &= 2\eta(X)g(Y, Z) - \eta(Y)g(X, Z) - \eta(Z)g(X, Y) \neq 0. \end{aligned}$$

$\bar{\nabla}$  defined by (3.1) satisfying (3.2) and (3.3) is a type of semi-symmetric non-metric connection.

### 4. CURVATURE TENSOR OF A KENMOTSU MANIFOLD WITH RESPECT TO THE SEMI-SYMMETRIC NON-METRIC CONNECTION

In this section we obtain the expressions of the curvature tensor, Ricci tensor and scalar curvature of  $M$  with respect to the semi-symmetric non-metric connection defined by (3.1).

Analogous to the definitions of the curvature tensor  $R$  of  $M$  with respect to the Levi-Civita connection  $\nabla$ , we define the curvature tensor  $\bar{R}$  of  $M$  with respect to the semi-symmetric non-metric connection  $\bar{\nabla}$  given by

$$(4.1) \quad \bar{R}(X, Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X, Y]}Z,$$

where  $X, Y, Z \in \chi(M)$ , the set of all differentiable vector fields on  $M$ .

Using (3.1) in (4.1), we get

$$(4.2) \quad \bar{R}(X, Y)Z = R(X, Y)Z - (\nabla_X \eta)(Y)Z + (\nabla_Y \eta)(X)Z - 2\eta(Y)g(X, Z)\xi + 2\eta(X)g(Y, Z)\xi + g(Y, Z)\nabla_X \xi - g(X, Z)\nabla_Y \xi.$$

Again using (2.5) and (2.6) in (4.2), we have

$$(4.3) \quad \bar{R}(X, Y)Z = R(X, Y)Z + g(Y, Z)X - g(X, Z)Y + \eta(X)g(Y, Z)\xi - \eta(Y)g(X, Z)\xi.$$

From (4.3), it follows that

$$(4.4) \quad \bar{R}(X, Y)Z = -\bar{R}(Y, X)Z,$$

and

$$(4.5) \quad \bar{R}(X, Y)Z + \bar{R}(Y, Z)X + \bar{R}(Z, X)Y = 0.$$

We call (4.5) the *first Bianchi identity* with respect to the semi-symmetric non-metric connection  $\bar{\nabla}$ .

Putting  $X = \xi$  in (4.3) and using (2.1) and (2.8), we obtain

$$(4.6) \quad \bar{R}(\xi, Y)Z = g(Y, Z)\xi - \eta(Y)\eta(Z)\xi,$$

where  $\xi \in \chi(M)$ .

Also putting  $Z = \xi$  in (4.3) and using (2.1) and (2.7), we get

$$(4.7) \quad \bar{R}(X, Y)\xi = 0.$$

Combining (2.9) and (4.3), we have

$$(4.8) \quad \eta(\bar{R}(X, Y)Z) = \eta(X)g(Y, Z) - \eta(Y)g(X, Z).$$

Taking the inner product of (4.3) with  $U$ , it follows that

$$(4.9) \quad \tilde{\bar{R}}(X, Y, Z, U) = \tilde{R}(X, Y, Z, U) + g(Y, Z)g(X, U) - g(X, Z)g(Y, U) + \eta(X)\eta(U)g(Y, Z) - \eta(Y)\eta(U)g(X, Z),$$

where  $U \in \chi(M)$ ,  $\tilde{\bar{R}}(X, Y, Z, U) = g(\bar{R}(X, Y)Z, U)$  and  $\tilde{R}(X, Y, Z, U) = g(R(X, Y)Z, U)$ .

From (4.9), we obtain

$$(4.10) \quad \tilde{\bar{R}}(X, Y, Z, U) = -\tilde{R}(X, Y, U, Z).$$

Let  $\{e_1, \dots, e_{2n+1}\}$  be a local orthonormal basis of the tangent space at a point of the manifold  $M$ . Then by putting  $X = U = e_i$  in (4.9) and taking summation over  $i$ ,  $1 \leq i \leq 2n+1$  and also using (2.1), we get

$$(4.11) \quad \bar{S}(Y, Z) = S(Y, Z) + (2n+1)g(Y, Z) - \eta(Y)\eta(Z),$$

where  $\bar{S}$  and  $S$  denote the Ricci tensor of  $M$  with respect to  $\bar{\nabla}$  and  $\nabla$  respectively.

From (4.11), we have

$$(4.12) \quad \bar{S}(Y, Z) = \bar{S}(Z, Y).$$

Putting  $Z = \xi$  in (4.11) and using (2.1) and (2.10), we obtain

$$(4.13) \quad \bar{S}(Y, \xi) = 0.$$

Let  $\bar{r}$  and  $r$  denote the scalar curvature of  $M$  with respect to  $\bar{\nabla}$  and  $\nabla$  respectively, i.e.,  $\bar{r} = \sum_{i=1}^{2n+1} \bar{S}(e_i, e_i)$  and  $r = \sum_{i=1}^{2n+1} S(e_i, e_i)$ . Again let  $\{e_1, \dots, e_{2n+1}\}$  be a local orthonormal basis of vector fields in  $M$ . Then by putting  $Y = Z = e_i$  in (4.11) and taking summation over  $i$ ,  $1 \leq i \leq 2n+1$  and also using (2.1), it follows that

$$(4.14) \quad \bar{r} = r + (2n+1)(2n+1) - 1.$$

Therefore, we can state the following :

**Proposition 4.1.** *For a Kenmotsu manifold  $M$  with respect to the semi-symmetric non-metric connection  $\bar{\nabla}$*

(i) *The curvature tensor  $\bar{R}$  is given by*

$$\bar{R}(X, Y)Z = R(X, Y)Z + g(Y, Z)X - g(X, Z)Y + \eta(X)g(Y, Z)\xi - \eta(Y)g(X, Z)\xi,$$

(ii) *The Ricci tensor  $\bar{S}$  is given by*

$$\bar{S}(Y, Z) = S(Y, Z) + (2n+1)g(Y, Z) - \eta(Y)\eta(Z),$$

(iii) *The scalar curvature  $\bar{r}$  is given by*

$$\bar{r} = r + (2n+1)(2n+1) - 1,$$

(iv)  $\bar{R}(X, Y)Z = -\bar{R}(Y, X)Z$ ,

(v)  $\bar{R}(X, Y)Z + \bar{R}(Y, Z)X + \bar{R}(Z, X)Y = 0$ ,

(vi) *The Ricci tensor  $\bar{S}$  is symmetric,*

(vii)  $\tilde{\bar{R}}(X, Y, Z, U) = -\tilde{R}(X, Y, U, Z)$ .

Now, we give an example of a 5-dimensional Kenmotsu manifold admitting a type of semi-symmetric non-metric connection which verify the skew-symmetric property and the first Bianchi identity of the curvature tensors  $\bar{R}$  of  $\bar{\nabla}$ .

**Example 4.2.** We consider the 5-dimensional manifold  $M = \{(x, y, z, u, v) \in \mathbb{R}^5\}$ , where  $(x, y, z, u, v)$  are the standard coordinates in  $\mathbb{R}^5$ .

We choose the vector fields

$$e_1 = e^{-v} \frac{\partial}{\partial x}, \quad e_2 = e^{-v} \frac{\partial}{\partial y}, \quad e_3 = e^{-v} \frac{\partial}{\partial z}, \quad e_4 = \frac{\partial}{\partial u}, \quad e_5 = \frac{\partial}{\partial v},$$

which are linearly independent at each point of  $M$ .

Let  $g$  be the Riemannian metric defined by

$$g(e_i, e_j) = 0, \quad i \neq j, \quad i, j = 1, 2, 3, 4, 5$$

and

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = g(e_4, e_4) = g(e_5, e_5) = 1.$$

Let  $\eta$  be the 1-form defined by

$$\eta(Z) = g(Z, e_5),$$

for any  $Z \in \chi(M)$ .

Let  $\phi$  be the  $(1, 1)$ -tensor field defined by

$$\phi e_1 = e_3, \quad \phi e_2 = e_4, \quad \phi e_3 = -e_1, \quad \phi e_4 = -e_2, \quad \phi e_5 = 0.$$

Using the linearity of  $\phi$  and  $g$ , we have

$$\begin{aligned} \eta(e_5) &= 1, \\ \phi^2(Z) &= -Z + \eta(Z)e_5 \end{aligned}$$

and

$$g(\phi Z, \phi U) = g(Z, U) - \eta(Z)\eta(U),$$

for any  $U, Z \in \chi(M)$ . Thus, for  $e_5 = \xi$ ,  $M(\phi, \xi, \eta, g)$  defines an almost contact metric manifold. The 1-form  $\eta$  is closed.

We have

$$\Omega\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right) = g\left(\frac{\partial}{\partial x}, \phi \frac{\partial}{\partial z}\right) = g\left(\frac{\partial}{\partial x}, -\frac{\partial}{\partial x}\right) = -e^{2v}.$$

Hence, we obtain  $\Omega = -e^{2v} dx \wedge dz$ . Thus,  $d\Omega = -2e^{2v} dv \wedge dx \wedge dz = 2\eta \wedge \Omega$ . Therefore,  $M(\phi, \xi, \eta, g)$  is an almost Kenmotsu manifold. It can be seen that  $M(\phi, \xi, \eta, g)$  is normal. So, it is a Kenmotsu manifold.

Then we have

$$\begin{aligned} [e_1, e_2] &= [e_1, e_3] = [e_1, e_4] = [e_2, e_3] = 0, \quad [e_1, e_5] = e_1, \\ [e_4, e_5] &= 0, \quad [e_2, e_4] = [e_3, e_4] = 0, \quad [e_2, e_5] = e_2, \quad [e_3, e_5] = e_3. \end{aligned}$$

The Levi-Civita connection  $\nabla$  of the metric tensor  $g$  is given by Koszul's formula which is given by

$$(4.15) \quad 2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).$$

Taking  $e_5 = \xi$  and using Koszul's formula we get the following

$$\begin{aligned} \nabla_{e_1} e_1 &= -e_5, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = 0, \quad \nabla_{e_1} e_4 = 0, \quad \nabla_{e_1} e_5 = e_1, \\ \nabla_{e_2} e_1 &= 0, \quad \nabla_{e_2} e_2 = -e_5, \quad \nabla_{e_2} e_3 = 0, \quad \nabla_{e_2} e_4 = 0, \quad \nabla_{e_2} e_5 = e_2, \\ \nabla_{e_3} e_1 &= 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = -e_5, \quad \nabla_{e_3} e_4 = 0, \quad \nabla_{e_3} e_5 = e_3, \\ \nabla_{e_4} e_1 &= 0, \quad \nabla_{e_4} e_2 = 0, \quad \nabla_{e_4} e_3 = 0, \quad \nabla_{e_4} e_4 = -e_5, \quad \nabla_{e_4} e_5 = e_4, \\ \nabla_{e_5} e_1 &= 0, \quad \nabla_{e_5} e_2 = 0, \quad \nabla_{e_5} e_3 = 0, \quad \nabla_{e_5} e_4 = 0, \quad \nabla_{e_5} e_5 = 0. \end{aligned}$$

Using (3.1) in the above equations, we obtain  
(4.16)

$$\begin{aligned}
\bar{\nabla}_{e_1} e_1 &= 0, \quad \bar{\nabla}_{e_1} e_2 = 0, \quad \bar{\nabla}_{e_1} e_3 = 0, \quad \bar{\nabla}_{e_1} e_4 = 0, \quad \bar{\nabla}_{e_1} e_5 = e_1, \\
\bar{\nabla}_{e_2} e_1 &= 0, \quad \bar{\nabla}_{e_2} e_2 = 0, \quad \bar{\nabla}_{e_2} e_3 = 0, \quad \bar{\nabla}_{e_2} e_4 = 0, \quad \bar{\nabla}_{e_2} e_5 = e_2, \\
\bar{\nabla}_{e_3} e_1 &= 0, \quad \bar{\nabla}_{e_3} e_2 = 0, \quad \bar{\nabla}_{e_3} e_3 = 0, \quad \bar{\nabla}_{e_3} e_4 = 0, \quad \bar{\nabla}_{e_3} e_5 = e_3, \\
\bar{\nabla}_{e_4} e_1 &= 0, \quad \bar{\nabla}_{e_4} e_2 = 0, \quad \bar{\nabla}_{e_4} e_3 = 0, \quad \bar{\nabla}_{e_4} e_4 = 0, \quad \bar{\nabla}_{e_4} e_5 = e_4, \\
\bar{\nabla}_{e_5} e_1 &= -e_1, \quad \bar{\nabla}_{e_5} e_2 = -e_2, \quad \bar{\nabla}_{e_5} e_3 = -e_3, \quad \bar{\nabla}_{e_5} e_4 = -e_4, \quad \bar{\nabla}_{e_5} e_5 = 0.
\end{aligned}$$

By the above results, we can easily obtain the non-vanishing components of the curvature tensors as follows:

$$\begin{aligned}
R(e_1, e_2)e_2 &= R(e_1, e_3)e_3 = R(e_1, e_4)e_4 = -e_1, \\
R(e_1, e_2)e_1 &= e_2, \quad R(e_1, e_3)e_1 = R(e_2, e_3)e_2 = e_3, \\
R(e_2, e_3)e_3 &= R(e_2, e_4)e_4 = R(e_2, e_5)e_5 = -e_2 \\
R(e_3, e_4)e_4 &= -e_3, \quad R(e_2, e_5)e_2 = R(e_1, e_5)e_1 = e_5, \\
R(e_1, e_4)e_1 &= R(e_2, e_4)e_2 = R(e_3, e_4)e_3 = e_4
\end{aligned}$$

and

$$\bar{R}(e_4, e_5)e_5 = e_4, \quad \bar{R}(e_5, e_4)e_5 = -e_4.$$

Let  $X, Y$  and  $Z$  be any three vector fields given by

$$X = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5, \quad Y = b_1 e_1 + b_2 e_2 + b_3 e_3 + b_4 e_4 + b_5 e_5$$

and  $Z = c_1 e_1 + c_2 e_2 + c_3 e_3 + c_4 e_4 + c_5 e_5$ , where  $a_i, b_i, c_i$ , for all  $i = 1, 2, 3, 4, 5$  are all non-zero real numbers.

Using (4.16), we obtain

$$\bar{R}(X, Y)Z = a_4 b_5 c_5 e_4 - a_5 b_4 c_5 e_4 = -(a_5 b_4 c_5 e_4 - a_4 b_5 c_5 e_4) = -\bar{R}(Y, X)Z.$$

Again using (4.16), we get

$$\bar{R}(X, Y)Z + \bar{R}(Y, Z)X + \bar{R}(Z, X)Y = (a_4 b_5 c_5 - a_5 b_4 c_5 + a_5 b_4 c_5 - a_5 b_5 c_4 + a_5 b_5 c_4 - a_4 b_5 c_5)e_4 = 0.$$

Hence the manifold under consideration satisfies the properties (iv) and (v) of Proposition 4.1.

## 5. THE CURVATURE TENSOR IS COVARIANT CONSTANT WITH RESPECT TO THE SEMI-SYMMETRIC NON-METRIC CONNECTION AND $M$ IS RECURRENT WITH RESPECT TO THE LEVI-CIVITA CONNECTION

In this section we characterize a Kenmotsu manifold admitting a type of semi-symmetric non-metric connection.

**Definition 5.1.** A Kenmotsu manifold  $M$  with respect to the Levi-Civita connection is said to be recurrent [23] if its curvature tensor  $R$  satisfies the condition

$$(5.1) \quad (\nabla_U R)(X, Y)Z = A(U)R(X, Y)Z,$$

where  $A$  is a non-zero 1-form and  $X, Y, Z, U \in \chi(M)$ .

Using (2.1), (2.7), (2.8) and (3.1), we obtain  
(5.2)

$$\begin{aligned} (\bar{\nabla}_U R)(X, Y)Z &= \bar{\nabla}_U R(X, Y)Z - R(\bar{\nabla}_U X, Y)Z - R(X, \bar{\nabla}_U Y)Z - R(X, Y)\bar{\nabla}_U Z \\ &= (\nabla_U R)(X, Y)Z + 2\eta(U)R(X, Y)Z + g(R(X, Y)Z, U)\xi + \eta(Z)g(U, Y)X - \eta(Z)g(U, X)Y \\ &\quad + \eta(Y)g(Z, U)X - \eta(X)g(Z, U)Y + g(X, U)g(Y, Z)\xi - g(X, Z)g(Y, U)\xi. \end{aligned}$$

Suppose  $(\bar{\nabla}_U R)(X, Y)Z = 0$ , then from (5.2), we get  
(5.3)

$$\begin{aligned} (\nabla_U R)(X, Y)Z + 2\eta(U)R(X, Y)Z + g(R(X, Y)Z, U)\xi + \eta(Z)g(U, Y)X - \eta(Z)g(U, X)Y + \\ \eta(Y)g(Z, U)X - \eta(X)g(Z, U)Y + g(X, U)g(Y, Z)\xi - g(X, Z)g(Y, U)\xi = 0. \end{aligned}$$

Using (5.1) in (5.3), we have  
(5.4)

$$\begin{aligned} A(U)R(X, Y)Z + 2\eta(U)R(X, Y)Z + g(R(X, Y)Z, U)\xi + \eta(Z)g(U, Y)X - \eta(Z)g(U, X)Y \\ + \eta(Y)g(Z, U)X - \eta(X)g(Z, U)Y + g(X, U)g(Y, Z)\xi - g(X, Z)g(Y, U)\xi = 0. \end{aligned}$$

Now, taking the inner product of (5.4) with  $\xi$  and using (2.1) and (2.9), it follows that  
(5.5)

$$\begin{aligned} A(U)\eta(Y)g(X, Z) - A(U)\eta(X)g(Y, Z) + 2\eta(U)\eta(Y)g(X, Z) - 2\eta(U)\eta(X)g(Y, Z) + g(R(X, Y)Z, U) \\ - \eta(Y)\eta(Z)g(X, U) + \eta(X)\eta(Z)g(Y, U) + g(X, U)g(Y, Z) - g(Y, U)g(X, Z) = 0. \end{aligned}$$

Contracting (5.5) over  $X$  and  $U$ , we obtain  
(5.6)

$$S(Y, Z) = 2(n-1)\eta(Y)\eta(Z) + [A(\xi) + 2 - 2n]g(Y, Z) - A(Z)\eta(Y).$$

Since the Ricci tensor  $S$  with respect to the Levi-Civita connection is symmetric; then from (5.6), we get  
(5.7)

$$A(Z)\eta(Y) = A(Y)\eta(Z).$$

Putting  $Y = \xi$  in (5.7) and using (2.1), we have  
(5.8)

$$A(Z) = A(\xi)\eta(Z).$$

Combining (5.6) and (5.8), it follows that  
(5.9)

$$S(Y, Z) = [2(n-1) - A(\xi)]\eta(Y)\eta(Z) + [A(\xi) + 2 - 2n]g(Y, Z),$$

where  $\lambda_1 = A(\xi) + 2 - 2n$ ,  $\lambda_2 = 2(n-1) - A(\xi)$ .

This result shows that the manifold is an  $\eta$ -Einstein manifold. The above discussion helps us to state the following theorem:

**Theorem 5.2.** *If in a Kenmotsu manifold the curvature tensor is covariant constant with respect to the semi-symmetric non-metric connection and the manifold is recurrent with respect to the Levi-Civita connection, then the manifold is an  $\eta$ -Einstein manifold.*

Let  $\xi^\perp$  denote the  $2n$ - dimensional distribution orthogonal to  $\xi$  in a Kenmotsu manifold admitting a semi-symmetric non-metric connection whose curvature tensor vanishes. Then for any  $X \in \xi^\perp$ ,  $g(X, \xi) = 0$  or,  $\eta(X) = 0$ . Now we shall determine the sectional curvature  $'R$  of the plane determine by the vectors  $X, Y \in \xi^\perp$ .

Putting  $\tilde{R} = 0$  and  $Z = Y$ ,  $U = X$  in (4.9), we get

$$\tilde{R}(X, Y, Y, X) = -[g(X, X)g(Y, Y) - g(X, Y)g(X, Y)].$$

Then

$${}'R(X, Y) = \frac{\tilde{R}(X, Y, Y, X)}{g(X, X)g(Y, Y) - g(X, Y)^2} = -1.$$

Summing up we can state the following theorem :

**Theorem 5.3.** *If in a Kenmotsu manifold the curvature tensor of the semi-symmetric non-metric connection vanishes, then the sectional curvature of the plane determined by two vectors  $X, Y \in \xi^\perp$  is  $-1$ .*

## 6. SEMI-SYMMETRIC KENMOTSU MANIFOLDS WITH RESPECT TO THE SEMI-SYMMETRIC NON-METRIC CONNECTION

In this section, we consider semi-symmetric Kenmotsu manifolds with respect to the semi-symmetric non-metric connection.

We suppose that the manifold under consideration is semi-symmetric with respect to the semi-symmetric non-metric connection  $\bar{\nabla}$ . Then

$$(\bar{R}(X, Y) \bullet \bar{R})(U, V)Z = 0.$$

So,

$$(6.1) \quad \bar{R}(X, Y)\bar{R}(U, V)Z - \bar{R}(\bar{R}(X, Y)U, V)Z - \bar{R}(U, \bar{R}(X, Y)V)Z - \bar{R}(U, V)\bar{R}(X, Y)Z = 0.$$

Replacing  $X$  by  $\xi$  in (6.1) and using (2.1), (4.6), (4.7) and (4.8), it follows that

$$(6.2) \quad g(\bar{R}(U, V)Z, Y)\xi - g(Y, U)g(V, Z)\xi + \eta(V)\eta(Z)g(Y, U)\xi + g(Y, V)g(U, Z)\xi - \eta(U)\eta(Z)g(Y, V)\xi = 0.$$

Now, taking the inner product of (6.2) with  $\xi$  and using (2.1), we obtain

$$(6.3) \quad g(\bar{R}(U, V)Z, Y) - g(Y, U)g(V, Z) + \eta(V)\eta(Z)g(Y, U) + g(Y, V)g(U, Z) - \eta(U)\eta(Z)g(Y, V) = 0.$$

Contracting  $U$  and  $Y$  in (6.3) and using (2.1) and (4.11), we have

$$S(V, Z) = -g(V, Z) + (1 - 2n)\eta(V)\eta(Z).$$

This result shows that the manifold is an  $\eta$ -Einstein manifold. Now, we are in a position to state the following theorem:

**Theorem 6.1.** *Let  $M$  be a semi-symmetric Kenmotsu manifold with respect to the semi-symmetric non-metric connection  $\bar{\nabla}$ . Then the manifold is an  $\eta$ -Einstein manifold with respect to the Levi-Civita connection.*

Now, using Lemma 2.1 we can state the following:

**Corollary 6.2.** *Let  $M$  be a semi-symmetric Kenmotsu manifold with respect to the semi-symmetric non-metric connection  $\bar{\nabla}$ . Then the manifold is an Einstein manifold with respect to the Levi-Civita connection.*

Since a symmetric manifold ( $\nabla R = 0$ ) implies semi-symmetric manifold ( $R \bullet R = 0$ ), therefore we obtain the following:

**Corollary 6.3.** *A symmetric (i.e.,  $\bar{\nabla} R = 0$ ) Kenmotsu manifold with respect to the semi-symmetric non-metric connection is an Einstein manifold.*

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## REFERENCES

- [1] N. S. Agashe and M. R. Chafle, *A semi-symmetric non-metric connection on a Riemannian Manifold*, Indian J. Pure Appl. Math., **23**(1992), 399-409.
- [2] K. Amur and S. S. Pujar, *On submanifolds of a Riemannian manifold admitting a metric semi-symmetric connection*, Tensor, N. S., **32**(1978), 35-38.
- [3] O. C. Andonie, *On semi-symmetric non-metric connection on a Riemannian manifold*, Ann. Fac. Sci. De Kinshasa, Zaire Sect. Math. Phys., **2**(1976).
- [4] B. Barua and S. Mukhopadhyay, *A sequence of semi-symmetric connections on a Riemannian manifold*, Proceedings of seventh national seminar on Finsler, Lagrange and Hamiltonian spaces, 1992, Brasov, Romania.
- [5] T. Q. Binh, *On semi-symmetric connections*, Periodica Math. Hungarica, **21**(1990), 101-107.
- [6] D.E. Blair, *Contact manifolds in Riemannian geometry*, Lecture Note in Mathematics, 509, Springer-Verlag Berlin, 1976.
- [7] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Birkhäuser, Boston, 2002.
- [8] U. C. De, *On a type of semi-symmetric connection on a Riemannian manifold*, Indian J. Pure Appl. Math., **21**(1990), 334-338.
- [9] U. C. De and S. C. Biswas, *On a type of semi-symmetric non-metric connection on a Riemannian manifold*, Ganita, **48**(1997), 91-94.
- [10] U. C. De and D. Kamilya, *Hypersurfaces of a Riemannian manifold with semi-symmetric non-metric connection*, J. Indian Inst. Sci., **75**(1995), 707-710.
- [11] G. Dileo and A. M. Pastore, *Almost Kenmotsu manifolds and local symmetry*, Bull. Belg. Math. Soc. Simon Stevin, **14**(2007), 343-354.
- [12] A. Friedman, A. and J. A. Schouten, *Über die Geometrie der halbsymmetrischen Übertragung*, Math., Zeitschr., **21**(1924), 211-223.
- [13] H. A. Hayden, *Subspaces of space with torsion*, Proc. London Math. Soc., **34**(1932), 27-50.
- [14] K. Kenmotsu, *A class of almost contact Riemannian manifolds*, Tohoku Math. J., **24**(1972), 93-103.
- [15] T. W. Kim and H. K. Pak, *Canonical foliations of certain classes of almost contact metric structures*, Acta Math. Sin. (Engl. Ser.), **21**(2005), 841-846.
- [16] D. Kowalczyk, *On some subclass of semisymmetric manifolds*, Soochow J. Math., **27**(2001), 445-461.
- [17] Y. Liang, *On semi-symmetric recurrent-metric connection*, Tensor, N. S., **55** (1994), 107-112.
- [18] Z. Olszak, *Locally conformal almost cosymplectic manifolds*, Colloq. Math., **57**(1989), 73-87.

- [19] C. Ozgur and S. Sular, *Warped product manifolds with semi-symmetric metric connections*, Taiwan. J. Math., **15**(2011), 1701-1719.
- [20] C. Ozgur and S. Sular, *Generalized Sasakian space forms with semi-symmetric metric connections*, An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N. S.), forthcoming.
- [21] Z. F. Ozen, S. A. Demirbag, S. A. Uysal and H. B. Yilmaz, *Some vector fields on a Riemannian manifold with semi-symmetric metric connection*, Bull. Iranian Math. Soc., **38**(2012), 479-490.
- [22] Z. F. Ozen, S. A. Uysal and S. A. Demirbag, *On sectional curvature of a Riemannian manifold with semi-symmetric metric connection*, Ann. Polon. Math., **101**(2011), 131-138.
- [23] E. M. Patterson, *Some theorems on Ricci-recurrent spaces*, Journal London Math. Soc., **27**(1952), 287-295.
- [24] M. Prvanovic, *On pseudo metric semi-symmetric connections*, Pub. De L' Institut Math., Nouvelle serie, **18**(1975), 157-164.
- [25] D. Smaranda, *Pseudo Riemannian recurrent manifolds with almost constant curvature*, The XVIII Int. conf. on Geometry and Topology (Oradea 1989), pp 88-2, Univ. "Babes Bolyai" Cluj-Napoca, 1988.
- [26] D. Smaranda and O. C. Andonie, *On semi-symmetric connections*, Ann. Fac. Sci. Univ. Nat. Zaire (Kinshasa), Sec. Math.-Phys., **2**(1976), 265-270.
- [27] R. N. Singh, *On a product semi-symmetric non-metric connection in a locally decomposable Riemannian manifold*, International Math. Forum, **6**(2011), 1893-1902.
- [28] R. N. Singh and M. K.Pandey, *On a type of semi-symmetric metric connection on a Riemannian manifold*, Bull. Calcutta Math. Soc., **16**(2008), 179-184.
- [29] R. N. Singh and G. Pandey, *On the  $W_2$ -curvature tensor of the semi-symmetric non-metric connection in a Kenmotsu manifold*, Navi Sad J. Math., **43**(2013), 91-105.
- [30] R. N. Singh and M. K.Pandey, *On semi-symmetric non-metric connection I*, Ganita, **58**(2007), 47-59.
- [31] Z. I. Szabo, *Structure theorems on Riemannian spaces satisfying  $R(X, Y) \cdot R = 0$ . I. The local version*, J. Diff. Geom., **17**(1982), 531-582.
- [32] Z. I. Szabo, *Classification and construction of complete hypersurfaces satisfying  $R(X, Y) \cdot R = 0$* , Acta Sci. Math.50, (Szeged), **47**(1981), 321-348.
- [33] Z. I. Szabo, *Structure theorems on Riemannian spaces satisfying  $R(X, Y) \cdot R = 0$ . II. Global version*, Geom. Dedicat., **19**(1985), 65-108.
- [34] S. Tanno, *The automorphism groups of almost contact Riemannian manifolds*, Tohoku Math. j., **21**(1969), 21-38.

- [35] K. Yano, *On semi-symmetric connection*, Revue Roumaine de Math. Pures et Appliques, **15**(1970), 1570-1586.

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