

SEMI-SYMMETRIC NON-METRIC CONNECTIONS ON KENMOTSU MANIFOLDS

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ABSTRACT. The object of the present paper is to study a type of semi-symmetric non-metric connection on a Kenmotsu manifold.

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1. INTRODUCTION

Manifolds known as Kenmotsu manifolds have been studied by K. Kenmotsu in 1972 [14]. They set up one of the three classes of almost contact Riemannian manifolds whose automorphism group attains the maximum dimension [34]. Consider an almost contact metric manifold M^{2n+1} , with structure (ϕ, ξ, η, g) given by a tensor field ϕ of type $(1, 1)$, a vector field ξ , a 1-form η satisfying $\phi^2 = -I + \eta \otimes \xi$, $\eta(\xi) = 1$, and a Riemannian metric g such that $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$ for any vector fields X and Y . The fundamental 2-form Φ is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields X and Y . The normality of an almost contact metric manifold is expressed by the vanishing of the tensor field $N = [\phi, \phi] + 2d\eta \otimes \xi$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ [6]. For more details we refer to Blair's books ([6],[7]). A Kenmotsu manifold can be defined as a normal almost contact metric manifold such that $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$. It is well known that Kenmotsu manifolds can be characterized, through their Levi-Civita connection, by $(\nabla_X \phi)(Y) = g(\phi X, Y)\xi - \eta(Y)\phi X$, for any vector fields X, Y, Z . Moreover Kenmotsu proved that such a manifold M^{2n+1} is locally a warped product $]-\varepsilon, \varepsilon[_f \times_f N^{2n}$, N^{2n} being a Kähler manifold and $f^2 = ce^{2t}$, c a positive constant.

More recently in ([15],[18]) and [11], almost contact metric manifolds such that η is closed and $d\Phi = 2\eta \wedge \Phi$ are studied and they are called *almost Kenmotsu*. Obviously a normal almost Kenmotsu manifold is a Kenmotsu manifold.

In 1924, Friedmann and Schouten [12] introduced the idea of semi-symmetric connection on a differentiable manifold. A linear connection $\bar{\nabla}$ on a differentiable manifold M is said to be a *semi-symmetric connection* if the torsion tensor T of the connection $\bar{\nabla}$ satisfies

$$(1.1) \quad T(X, Y) = u(Y)X - u(X)Y,$$

where u is a 1-form and ρ is a vector field defined by

$$(1.2) \quad u(X) = g(X, \rho),$$

for all vector fields $X \in \chi(M)$, $\chi(M)$ is the set of all differentiable vector fields on M .

In 1932, Hayden [13] introduced the idea of semi-symmetric metric connections on a Riemannian manifold (M, g) . A semi-symmetric connection $\tilde{\nabla}$ is said to be a *semi-symmetric metric connection* if

$$(1.3) \quad \tilde{\nabla}g = 0.$$

A relation between the semi-symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ of (M, g) was given by Yano [35]: $\tilde{\nabla}_X Y = \nabla_X Y + u(Y)X - g(X, Y)\rho$, where $u(X) = g(X, \rho)$.

The study of semi-symmetric metric connection was further developed by Amur and Pujar [2], Binh [5], De [8], Singh et al. [28], Ozgur et al ([19],[20]), Ozen, Uysal and Demirbag [22] and many others.

After a long gap the study of a semi-symmetric connection $\bar{\nabla}$ satisfying

$$(1.4) \quad \bar{\nabla}g \neq 0.$$

was initiated by Prvanović [24] with the name pseudo-metric semi-symmetric connection and was just followed by Andonie [3].

A semi-symmetric connection $\bar{\nabla}$ is said to be a *semi-symmetric non-metric connection* if it satisfies the condition (1.4).

In 1992, Agashe and Chafle [1] studied a semi-symmetric non-metric connection $\bar{\nabla}$, whose torsion tensor T satisfies $T(X, Y) = u(Y)X - u(X)Y$ and $(\bar{\nabla}_X g)(Y, Z) = -u(Y)g(X, Z) - u(Z)g(X, Y)$. They proved that the projective curvature tensor of the manifold with respect to these two connections are equal to each other. In 1992, Barua and Mukhopadhyay [4] studied a type of semi-symmetric connection $\bar{\nabla}$ which satisfies $(\bar{\nabla}_X g)(Y, Z) = 2u(X)g(Y, Z) - u(Y)g(X, Z) - u(Z)g(X, Y)$. Since $\bar{\nabla}g \neq 0$, this is another type of semi-symmetric non-metric connection. However, the authors preferred the name *semi-symmetric semimetric connection*.

In 1994, Liang [17] studied another type of semi-symmetric non-metric connection $\bar{\nabla}$ for which we have $(\bar{\nabla}_X g)(Y, Z) = 2u(X)g(Y, Z)$, where u is a non-zero 1-form and he called this a *semi-symmetric recurrent metric connection*.

The semi-symmetric non-metric connections was further developed by several authors such as De and Biswas [9], De and Kamilya [10], Liang [17], Singh et al. ([27], [29], [30]), Ozen, Demirbag, Ussal and Yilmaz [21], Smaranda [25], Smaranda and Andonie [26] and many others

A Riemannian manifold is said to be *semi-symmetric* ([31], [32], [33], [16]) if the curvature tensor R with respect to the Riemannian connection satisfies the condition

$$R(X, Y) \bullet R = 0,$$

where $R(X, Y)$ denotes the derivation of the tensor algebra at each point of the manifold for tangent vectors X, Y .

Semi-symmetric manifolds are the generalization of manifolds of constant curvature and locally symmetric manifolds (i.e., $\nabla R = 0$). Also Ricci semi-symmetric manifolds are the generalizations of semi-symmetric and Einstein manifolds.

In this paper we study a type of semi-symmetric non-metric connection due to Barua and Mukhopadhyay [4] on Kenmotsu manifolds. The paper is organized as follows : After introduction in Section 2, we

give a brief account of the Kenmotsu manifolds. In Section 3, we define a semi-symmetric non-metric connection. Section 4 deals with the relation of the curvature tensor between the Levi-Civita connection and the semi-symmetric non-metric connection on a Kenmotsu manifold and we also construct an example of a 5-dimensional Kenmotsu manifold admitting a type of semi-symmetric non-metric connection whose curvature tensor satisfies the skew-symmetric property and the first Bianchi identity. A Kenmotsu manifold whose curvature tensor is covariant constant with respect to the semi-symmetric non-metric connection and M is recurrent with respect to the Levi-Civita connection have been studied in Section 5. Finally, we have classified Kenmotsu manifolds which satisfy $\bar{R} \bullet \bar{R} = 0$ with respect to the semi-symmetric non-metric connection, where \bar{R} is the curvature tensor with respect to the semi-symmetric non-metric connection.

2. KENMOTSU MANIFOLDS

Let M be an $(2n + 1)$ -dimensional almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g) consisting of a $(1, 1)$ tensor field ϕ , a vector field ξ , a 1-form η and the Riemannian metric g on M satisfying [6]

$$(2.1) \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta(\phi(X)) = 0, \quad g(X, \xi) = \eta(X),$$

$$(2.2) \quad \phi^2(X) = -X + \eta(X)\xi,$$

$$(2.3) \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for all vector fields X, Y on $\chi(M)$. A manifold with the almost contact metric structure (ϕ, ξ, η, g) is an almost Kenmotsu manifold if the following conditions are satisfied

$$d\eta = 0; \quad d\Omega = 2\eta \wedge \Omega,$$

where Ω being the 2-form defined by $\Omega(X, Y) = g(X, \phi Y)$. Any normal almost Kenmotsu manifold is a Kenmotsu manifold. An almost contact metric structure (ϕ, ξ, η, g) is a Kenmotsu manifold [14] if and only if

$$(2.4) \quad (\nabla_X \phi)(Y) = g(\phi X, Y)\xi - \eta(Y)\phi X.$$

Hereafter we denote the Kenmotsu manifold of dimension $(2n + 1)$ by M . From the above relations, it follows that

$$(2.5) \quad \nabla_X \xi = X - \eta(X)\xi,$$

$$(2.6) \quad (\nabla_X \eta)(Y) = g(X, Y) - \eta(X)\eta(Y),$$

$$(2.7) \quad R(X, Y)\xi = \eta(X)Y - \eta(Y)X,$$

$$(2.8) \quad R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi,$$

$$(2.9) \quad \eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X),$$

$$(2.10) \quad S(X, \xi) = -2n\eta(X),$$

where R and S denote the curvature tensor and the Ricci tensor of M , respectively, with respect to the Levi-Civita connection.

Let M be a Kenmotsu manifold. M is said to be a η -Einstein manifold if there exists the real valued functions λ_1, λ_2 such that

$$S(X, Y) = \lambda_1 g(X, Y) + \lambda_2 \eta(X)\eta(Y).$$

For $\lambda_2 = 0$, the manifold M is an Einstein manifold.

Now we state the following:

Lemma 2.1. [14] *Let M be an η -Einstein Kenmotsu manifold of the form $S(X, Y) = \lambda_1 g(X, Y) + \lambda_2 \eta(X)\eta(Y)$. If $\lambda_2 = \text{constant}$ (or, $\lambda_1 = \text{constant}$), then M is an Einstein one.*

3. SEMI-SYMMETRIC NON-METRIC CONNECTION

This section deals with a type of semi-symmetric non-metric connection on a Kenmotsu manifold. We consider a type of semi-symmetric non-metric connection which was introduced by Barua and Mukhopadhyay [4]. In a Kenmotsu manifold a semi-symmetric non-metric connection is defined by $T(X, Y) = \eta(Y)X - \eta(X)Y$, with ξ as the associated vector field, i.e., $g(X, \xi) = \eta(X)$.

A relation between semi-symmetric non-metric connection $\bar{\nabla}$ and the Levi-Civita connection ∇ have been obtained by Barua and Mukhopadhyay [4] and is given by

$$(3.1) \quad \bar{\nabla}_X Y = \nabla_X Y - \eta(X)Y + g(X, Y)\xi.$$

Using (3.1), the torsion tensor T of M with respect to the connection $\bar{\nabla}$ is given by

$$(3.2) \quad T(X, Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X, Y] = \eta(Y)X - \eta(X)Y.$$

Hence a relation satisfying (3.2) is called a semi-symmetric connection.

Further using (3.1), we have

$$(3.3) \quad \begin{aligned} (\bar{\nabla}_X g)(Y, Z) &= \nabla_X g(Y, Z) - g(\bar{\nabla}_X Y, Z) - g(Y, \bar{\nabla}_X Z) \\ &= 2\eta(X)g(Y, Z) - \eta(Y)g(X, Z) - \eta(Z)g(X, Y) \neq 0. \end{aligned}$$

$\bar{\nabla}$ defined by (3.1) satisfying (3.2) and (3.3) is a type of semi-symmetric non-metric connection.

4. CURVATURE TENSOR OF A KENMOTSU MANIFOLD WITH RESPECT TO THE SEMI-SYMMETRIC NON-METRIC CONNECTION

In this section we obtain the expressions of the curvature tensor, Ricci tensor and scalar curvature of M with respect to the semi-symmetric non-metric connection defined by (3.1).

Analogous to the definitions of the curvature tensor R of M with respect to the Levi-Civita connection ∇ , we define the curvature tensor \bar{R} of M with respect to the semi-symmetric non-metric connection $\bar{\nabla}$ given by

$$(4.1) \quad \bar{R}(X, Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X, Y]}Z,$$

where $X, Y, Z \in \chi(M)$, the set of all differentiable vector fields on M .

Using (3.1) in (4.1), we get

$$(4.2) \quad \bar{R}(X, Y)Z = R(X, Y)Z - (\nabla_X \eta)(Y)Z + (\nabla_Y \eta)(X)Z - 2\eta(Y)g(X, Z)\xi + 2\eta(X)g(Y, Z)\xi + g(Y, Z)\nabla_X \xi - g(X, Z)\nabla_Y \xi.$$

Again using (2.5) and (2.6) in (4.2), we have

$$(4.3) \quad \bar{R}(X, Y)Z = R(X, Y)Z + g(Y, Z)X - g(X, Z)Y + \eta(X)g(Y, Z)\xi - \eta(Y)g(X, Z)\xi.$$

From (4.3), it follows that

$$(4.4) \quad \bar{R}(X, Y)Z = -\bar{R}(Y, X)Z,$$

and

$$(4.5) \quad \bar{R}(X, Y)Z + \bar{R}(Y, Z)X + \bar{R}(Z, X)Y = 0.$$

We call (4.5) the *first Bianchi identity* with respect to the semi-symmetric non-metric connection $\bar{\nabla}$.

Putting $X = \xi$ in (4.3) and using (2.1) and (2.8), we obtain

$$(4.6) \quad \bar{R}(\xi, Y)Z = g(Y, Z)\xi - \eta(Y)\eta(Z)\xi,$$

where $\xi \in \chi(M)$.

Also putting $Z = \xi$ in (4.3) and using (2.1) and (2.7), we get

$$(4.7) \quad \bar{R}(X, Y)\xi = 0.$$

Combining (2.9) and (4.3), we have

$$(4.8) \quad \eta(\bar{R}(X, Y)Z) = \eta(X)g(Y, Z) - \eta(Y)g(X, Z).$$

Taking the inner product of (4.3) with U , it follows that

$$(4.9) \quad \tilde{\bar{R}}(X, Y, Z, U) = \tilde{R}(X, Y, Z, U) + g(Y, Z)g(X, U) - g(X, Z)g(Y, U) + \eta(X)\eta(U)g(Y, Z) - \eta(Y)\eta(U)g(X, Z),$$

where $U \in \chi(M)$, $\tilde{\bar{R}}(X, Y, Z, U) = g(\bar{R}(X, Y)Z, U)$ and $\tilde{R}(X, Y, Z, U) = g(R(X, Y)Z, U)$.

From (4.9), we obtain

$$(4.10) \quad \tilde{\bar{R}}(X, Y, Z, U) = -\tilde{R}(X, Y, U, Z).$$

Let $\{e_1, \dots, e_{2n+1}\}$ be a local orthonormal basis of the tangent space at a point of the manifold M . Then by putting $X = U = e_i$ in (4.9) and taking summation over i , $1 \leq i \leq 2n+1$ and also using (2.1), we get

$$(4.11) \quad \bar{S}(Y, Z) = S(Y, Z) + (2n+1)g(Y, Z) - \eta(Y)\eta(Z),$$

where \bar{S} and S denote the Ricci tensor of M with respect to $\bar{\nabla}$ and ∇ respectively.

From (4.11), we have

$$(4.12) \quad \bar{S}(Y, Z) = \bar{S}(Z, Y).$$

Putting $Z = \xi$ in (4.11) and using (2.1) and (2.10), we obtain

$$(4.13) \quad \bar{S}(Y, \xi) = 0.$$

Let \bar{r} and r denote the scalar curvature of M with respect to $\bar{\nabla}$ and ∇ respectively, i.e., $\bar{r} = \sum_{i=1}^{2n+1} \bar{S}(e_i, e_i)$ and $r = \sum_{i=1}^{2n+1} S(e_i, e_i)$. Again let $\{e_1, \dots, e_{2n+1}\}$ be a local orthonormal basis of vector fields in M . Then by putting $Y = Z = e_i$ in (4.11) and taking summation over i , $1 \leq i \leq 2n+1$ and also using (2.1), it follows that

$$(4.14) \quad \bar{r} = r + (2n+1)(2n+1) - 1.$$

Therefore, we can state the following :

Proposition 4.1. *For a Kenmotsu manifold M with respect to the semi-symmetric non-metric connection $\bar{\nabla}$*

(i) *The curvature tensor \bar{R} is given by*

$$\bar{R}(X, Y)Z = R(X, Y)Z + g(Y, Z)X - g(X, Z)Y + \eta(X)g(Y, Z)\xi - \eta(Y)g(X, Z)\xi,$$

(ii) *The Ricci tensor \bar{S} is given by*

$$\bar{S}(Y, Z) = S(Y, Z) + (2n+1)g(Y, Z) - \eta(Y)\eta(Z),$$

(iii) *The scalar curvature \bar{r} is given by*

$$\bar{r} = r + (2n+1)(2n+1) - 1,$$

(iv) $\bar{R}(X, Y)Z = -\bar{R}(Y, X)Z$,

(v) $\bar{R}(X, Y)Z + \bar{R}(Y, Z)X + \bar{R}(Z, X)Y = 0$,

(vi) *The Ricci tensor \bar{S} is symmetric,*

(vii) $\tilde{\bar{R}}(X, Y, Z, U) = -\tilde{R}(X, Y, U, Z)$.

Now, we give an example of a 5-dimensional Kenmotsu manifold admitting a type of semi-symmetric non-metric connection which verify the skew-symmetric property and the first Bianchi identity of the curvature tensors \bar{R} of $\bar{\nabla}$.

Example 4.2. We consider the 5-dimensional manifold $M = \{(x, y, z, u, v) \in \mathbb{R}^5\}$, where (x, y, z, u, v) are the standard coordinates in \mathbb{R}^5 .

We choose the vector fields

$$e_1 = e^{-v} \frac{\partial}{\partial x}, \quad e_2 = e^{-v} \frac{\partial}{\partial y}, \quad e_3 = e^{-v} \frac{\partial}{\partial z}, \quad e_4 = \frac{\partial}{\partial u}, \quad e_5 = \frac{\partial}{\partial v},$$

which are linearly independent at each point of M .

Let g be the Riemannian metric defined by

$$g(e_i, e_j) = 0, \quad i \neq j, \quad i, j = 1, 2, 3, 4, 5$$

and

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = g(e_4, e_4) = g(e_5, e_5) = 1.$$

Let η be the 1-form defined by

$$\eta(Z) = g(Z, e_5),$$

for any $Z \in \chi(M)$.

Let ϕ be the $(1, 1)$ -tensor field defined by

$$\phi e_1 = e_3, \quad \phi e_2 = e_4, \quad \phi e_3 = -e_1, \quad \phi e_4 = -e_2, \quad \phi e_5 = 0.$$

Using the linearity of ϕ and g , we have

$$\begin{aligned} \eta(e_5) &= 1, \\ \phi^2(Z) &= -Z + \eta(Z)e_5 \end{aligned}$$

and

$$g(\phi Z, \phi U) = g(Z, U) - \eta(Z)\eta(U),$$

for any $U, Z \in \chi(M)$. Thus, for $e_5 = \xi$, $M(\phi, \xi, \eta, g)$ defines an almost contact metric manifold. The 1-form η is closed.

We have

$$\Omega\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right) = g\left(\frac{\partial}{\partial x}, \phi \frac{\partial}{\partial z}\right) = g\left(\frac{\partial}{\partial x}, -\frac{\partial}{\partial x}\right) = -e^{2v}.$$

Hence, we obtain $\Omega = -e^{2v} dx \wedge dz$. Thus, $d\Omega = -2e^{2v} dv \wedge dx \wedge dz = 2\eta \wedge \Omega$. Therefore, $M(\phi, \xi, \eta, g)$ is an almost Kenmotsu manifold. It can be seen that $M(\phi, \xi, \eta, g)$ is normal. So, it is a Kenmotsu manifold.

Then we have

$$\begin{aligned} [e_1, e_2] &= [e_1, e_3] = [e_1, e_4] = [e_2, e_3] = 0, \quad [e_1, e_5] = e_1, \\ [e_4, e_5] &= 0, \quad [e_2, e_4] = [e_3, e_4] = 0, \quad [e_2, e_5] = e_2, \quad [e_3, e_5] = e_3. \end{aligned}$$

The Levi-Civita connection ∇ of the metric tensor g is given by Koszul's formula which is given by

$$(4.15) \quad 2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).$$

Taking $e_5 = \xi$ and using Koszul's formula we get the following

$$\begin{aligned} \nabla_{e_1} e_1 &= -e_5, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = 0, \quad \nabla_{e_1} e_4 = 0, \quad \nabla_{e_1} e_5 = e_1, \\ \nabla_{e_2} e_1 &= 0, \quad \nabla_{e_2} e_2 = -e_5, \quad \nabla_{e_2} e_3 = 0, \quad \nabla_{e_2} e_4 = 0, \quad \nabla_{e_2} e_5 = e_2, \\ \nabla_{e_3} e_1 &= 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = -e_5, \quad \nabla_{e_3} e_4 = 0, \quad \nabla_{e_3} e_5 = e_3, \\ \nabla_{e_4} e_1 &= 0, \quad \nabla_{e_4} e_2 = 0, \quad \nabla_{e_4} e_3 = 0, \quad \nabla_{e_4} e_4 = -e_5, \quad \nabla_{e_4} e_5 = e_4, \\ \nabla_{e_5} e_1 &= 0, \quad \nabla_{e_5} e_2 = 0, \quad \nabla_{e_5} e_3 = 0, \quad \nabla_{e_5} e_4 = 0, \quad \nabla_{e_5} e_5 = 0. \end{aligned}$$

Using (3.1) in the above equations, we obtain
(4.16)

$$\begin{aligned}
\bar{\nabla}_{e_1} e_1 &= 0, \quad \bar{\nabla}_{e_1} e_2 = 0, \quad \bar{\nabla}_{e_1} e_3 = 0, \quad \bar{\nabla}_{e_1} e_4 = 0, \quad \bar{\nabla}_{e_1} e_5 = e_1, \\
\bar{\nabla}_{e_2} e_1 &= 0, \quad \bar{\nabla}_{e_2} e_2 = 0, \quad \bar{\nabla}_{e_2} e_3 = 0, \quad \bar{\nabla}_{e_2} e_4 = 0, \quad \bar{\nabla}_{e_2} e_5 = e_2, \\
\bar{\nabla}_{e_3} e_1 &= 0, \quad \bar{\nabla}_{e_3} e_2 = 0, \quad \bar{\nabla}_{e_3} e_3 = 0, \quad \bar{\nabla}_{e_3} e_4 = 0, \quad \bar{\nabla}_{e_3} e_5 = e_3, \\
\bar{\nabla}_{e_4} e_1 &= 0, \quad \bar{\nabla}_{e_4} e_2 = 0, \quad \bar{\nabla}_{e_4} e_3 = 0, \quad \bar{\nabla}_{e_4} e_4 = 0, \quad \bar{\nabla}_{e_4} e_5 = e_4, \\
\bar{\nabla}_{e_5} e_1 &= -e_1, \quad \bar{\nabla}_{e_5} e_2 = -e_2, \quad \bar{\nabla}_{e_5} e_3 = -e_3, \quad \bar{\nabla}_{e_5} e_4 = -e_4, \quad \bar{\nabla}_{e_5} e_5 = 0.
\end{aligned}$$

By the above results, we can easily obtain the non-vanishing components of the curvature tensors as follows:

$$\begin{aligned}
R(e_1, e_2)e_2 &= R(e_1, e_3)e_3 = R(e_1, e_4)e_4 = -e_1, \\
R(e_1, e_2)e_1 &= e_2, \quad R(e_1, e_3)e_1 = R(e_2, e_3)e_2 = e_3, \\
R(e_2, e_3)e_3 &= R(e_2, e_4)e_4 = R(e_2, e_5)e_5 = -e_2 \\
R(e_3, e_4)e_4 &= -e_3, \quad R(e_2, e_5)e_2 = R(e_1, e_5)e_1 = e_5, \\
R(e_1, e_4)e_1 &= R(e_2, e_4)e_2 = R(e_3, e_4)e_3 = e_4
\end{aligned}$$

and

$$\bar{R}(e_4, e_5)e_5 = e_4, \quad \bar{R}(e_5, e_4)e_5 = -e_4.$$

Let X, Y and Z be any three vector fields given by

$$X = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5, \quad Y = b_1 e_1 + b_2 e_2 + b_3 e_3 + b_4 e_4 + b_5 e_5$$

and $Z = c_1 e_1 + c_2 e_2 + c_3 e_3 + c_4 e_4 + c_5 e_5$, where a_i, b_i, c_i , for all $i = 1, 2, 3, 4, 5$ are all non-zero real numbers.

Using (4.16), we obtain

$$\bar{R}(X, Y)Z = a_4 b_5 c_5 e_4 - a_5 b_4 c_5 e_4 = -(a_5 b_4 c_5 e_4 - a_4 b_5 c_5 e_4) = -\bar{R}(Y, X)Z.$$

Again using (4.16), we get

$$\bar{R}(X, Y)Z + \bar{R}(Y, Z)X + \bar{R}(Z, X)Y = (a_4 b_5 c_5 - a_5 b_4 c_5 + a_5 b_4 c_5 - a_5 b_5 c_4 + a_5 b_5 c_4 - a_4 b_5 c_5)e_4 = 0.$$

Hence the manifold under consideration satisfies the properties (iv) and (v) of Proposition 4.1.

5. THE CURVATURE TENSOR IS COVARIANT CONSTANT WITH RESPECT TO THE SEMI-SYMMETRIC NON-METRIC CONNECTION AND M IS RECURRENT WITH RESPECT TO THE LEVI-CIVITA CONNECTION

In this section we characterize a Kenmotsu manifold admitting a type of semi-symmetric non-metric connection.

Definition 5.1. A Kenmotsu manifold M with respect to the Levi-Civita connection is said to be recurrent [23] if its curvature tensor R satisfies the condition

$$(5.1) \quad (\nabla_U R)(X, Y)Z = A(U)R(X, Y)Z,$$

where A is a non-zero 1-form and $X, Y, Z, U \in \chi(M)$.

Using (2.1), (2.7), (2.8) and (3.1), we obtain
(5.2)

$$\begin{aligned} (\bar{\nabla}_U R)(X, Y)Z &= \bar{\nabla}_U R(X, Y)Z - R(\bar{\nabla}_U X, Y)Z - R(X, \bar{\nabla}_U Y)Z - R(X, Y)\bar{\nabla}_U Z \\ &= (\nabla_U R)(X, Y)Z + 2\eta(U)R(X, Y)Z + g(R(X, Y)Z, U)\xi + \eta(Z)g(U, Y)X - \eta(Z)g(U, X)Y \\ &\quad + \eta(Y)g(Z, U)X - \eta(X)g(Z, U)Y + g(X, U)g(Y, Z)\xi - g(X, Z)g(Y, U)\xi. \end{aligned}$$

Suppose $(\bar{\nabla}_U R)(X, Y)Z = 0$, then from (5.2), we get
(5.3)

$$\begin{aligned} (\nabla_U R)(X, Y)Z + 2\eta(U)R(X, Y)Z + g(R(X, Y)Z, U)\xi + \eta(Z)g(U, Y)X - \eta(Z)g(U, X)Y + \\ \eta(Y)g(Z, U)X - \eta(X)g(Z, U)Y + g(X, U)g(Y, Z)\xi - g(X, Z)g(Y, U)\xi = 0. \end{aligned}$$

Using (5.1) in (5.3), we have
(5.4)

$$\begin{aligned} A(U)R(X, Y)Z + 2\eta(U)R(X, Y)Z + g(R(X, Y)Z, U)\xi + \eta(Z)g(U, Y)X - \eta(Z)g(U, X)Y \\ + \eta(Y)g(Z, U)X - \eta(X)g(Z, U)Y + g(X, U)g(Y, Z)\xi - g(X, Z)g(Y, U)\xi = 0. \end{aligned}$$

Now, taking the inner product of (5.4) with ξ and using (2.1) and (2.9), it follows that
(5.5)

$$\begin{aligned} A(U)\eta(Y)g(X, Z) - A(U)\eta(X)g(Y, Z) + 2\eta(U)\eta(Y)g(X, Z) - 2\eta(U)\eta(X)g(Y, Z) + g(R(X, Y)Z, U) \\ - \eta(Y)\eta(Z)g(X, U) + \eta(X)\eta(Z)g(Y, U) + g(X, U)g(Y, Z) - g(Y, U)g(X, Z) = 0. \end{aligned}$$

Contracting (5.5) over X and U , we obtain
(5.6)

$$S(Y, Z) = 2(n-1)\eta(Y)\eta(Z) + [A(\xi) + 2 - 2n]g(Y, Z) - A(Z)\eta(Y).$$

Since the Ricci tensor S with respect to the Levi-Civita connection is symmetric; then from (5.6), we get

$$(5.7) \quad A(Z)\eta(Y) = A(Y)\eta(Z).$$

Putting $Y = \xi$ in (5.7) and using (2.1), we have
(5.8)

$$A(Z) = A(\xi)\eta(Z).$$

Combining (5.6) and (5.8), it follows that
(5.9)

$$S(Y, Z) = [2(n-1) - A(\xi)]\eta(Y)\eta(Z) + [A(\xi) + 2 - 2n]g(Y, Z),$$

where $\lambda_1 = A(\xi) + 2 - 2n$, $\lambda_2 = 2(n-1) - A(\xi)$.

This result shows that the manifold is an η -Einstein manifold. The above discussion helps us to state the following theorem:

Theorem 5.2. *If in a Kenmotsu manifold the curvature tensor is covariant constant with respect to the semi-symmetric non-metric connection and the manifold is recurrent with respect to the Levi-Civita connection, then the manifold is an η -Einstein manifold.*

Let ξ^\perp denote the $2n$ - dimensional distribution orthogonal to ξ in a Kenmotsu manifold admitting a semi-symmetric non-metric connection whose curvature tensor vanishes. Then for any $X \in \xi^\perp$, $g(X, \xi) = 0$ or, $\eta(X) = 0$. Now we shall determine the sectional curvature $'R$ of the plane determine by the vectors $X, Y \in \xi^\perp$.

Putting $\tilde{R} = 0$ and $Z = Y, U = X$ in (4.9), we get

$$\tilde{R}(X, Y, Y, X) = -[g(X, X)g(Y, Y) - g(X, Y)g(X, Y)].$$

Then

$${}'R(X, Y) = \frac{\tilde{R}(X, Y, Y, X)}{g(X, X)g(Y, Y) - g(X, Y)^2} = -1.$$

Summing up we can state the following theorem :

Theorem 5.3. *If in a Kenmotsu manifold the curvature tensor of the semi-symmetric non-metric connection vanishes, then the sectional curvature of the plane determined by two vectors $X, Y \in \xi^\perp$ is -1 .*

6. SEMI-SYMMETRIC KENMOTSU MANIFOLDS WITH RESPECT TO THE SEMI-SYMMETRIC NON-METRIC CONNECTION

In this section, we consider semi-symmetric Kenmotsu manifolds with respect to the semi-symmetric non-metric connection.

We suppose that the manifold under consideration is semi-symmetric with respect to the semi-symmetric non-metric connection $\bar{\nabla}$. Then

$$(\bar{R}(X, Y) \bullet \bar{R})(U, V)Z = 0.$$

So,

$$(6.1) \quad \bar{R}(X, Y)\bar{R}(U, V)Z - \bar{R}(\bar{R}(X, Y)U, V)Z - \bar{R}(U, \bar{R}(X, Y)V)Z - \bar{R}(U, V)\bar{R}(X, Y)Z = 0.$$

Replacing X by ξ in (6.1) and using (2.1), (4.6), (4.7) and (4.8), it follows that

$$(6.2) \quad g(\bar{R}(U, V)Z, Y)\xi - g(Y, U)g(V, Z)\xi + \eta(V)\eta(Z)g(Y, U)\xi + g(Y, V)g(U, Z)\xi - \eta(U)\eta(Z)g(Y, V)\xi = 0.$$

Now, taking the inner product of (6.2) with ξ and using (2.1), we obtain

$$(6.3) \quad g(\bar{R}(U, V)Z, Y) - g(Y, U)g(V, Z) + \eta(V)\eta(Z)g(Y, U) + g(Y, V)g(U, Z) - \eta(U)\eta(Z)g(Y, V) = 0.$$

Contracting U and Y in (6.3) and using (2.1) and (4.11), we have

$$S(V, Z) = -g(V, Z) + (1 - 2n)\eta(V)\eta(Z).$$

This result shows that the manifold is an η -Einstein manifold. Now, we are in a position to state the following theorem:

Theorem 6.1. *Let M be a semi-symmetric Kenmotsu manifold with respect to the semi-symmetric non-metric connection $\bar{\nabla}$. Then the manifold is an η -Einstein manifold with respect to the Levi-Civita connection.*

Now, using Lemma 2.1 we can state the following:

Corollary 6.2. *Let M be a semi-symmetric Kenmotsu manifold with respect to the semi-symmetric non-metric connection $\bar{\nabla}$. Then the manifold is an Einstein manifold with respect to the Levi-Civita connection.*

Since a symmetric manifold ($\nabla R = 0$) implies semi-symmetric manifold ($R \bullet R = 0$), therefore we obtain the following:

Corollary 6.3. *A symmetric (i.e., $\bar{\nabla}R = 0$) Kenmotsu manifold with respect to the semi-symmetric non-metric connection is an Einstein manifold.*

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