ON A GENERALIZED GAUSS CONVERGENCE CRITERION

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Abstract. In this paper we combine the well known Raabe-Duhamel, Kummer, Bertrand ... criterions of convergence for series with positive terms and we obtain a new one which is more powerful than those cited before. Even the famous Gauss criterion, which was in fact our starting point, is a consequence of this new convergence test.

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1. Preliminary and first results

It is well known that the sequence $((1+\frac{1}{n})^n)_{n\geq 1}$ (resp. $((1+\frac{1}{n})^{n+1})_{n\geq 1})$ increases (respectively decreases) to the real number $e = 2,73\ldots$.

Sometimes is useful to rephrase this assertion in a more powerful form:

Lemma 1. *The function f (resp. g) defined by*

$$f(x) = \left(1+\frac{1}{x}\right)^x \quad (\text{resp. } g(x) = \left(1+\frac{1}{x}\right)^{x+1}), \quad x \in (0,\infty)$$

is increasing (resp. decreasing) to the real number e when x tends to ∞.

Notations. The function $E_1 : (-\infty,\infty) \to (0,\infty)$ given by $E_1(x) = e^x$ being increasing, bijective and continuous (in fact $E_1 \in \mathcal{C}^\infty(\mathbb{R})$), the functions E_p, $p \in \mathbb{N}^*$, inductively defined by

$$E_{p+1}(x) = E_1(E_p(x)) = E_p(E_1(x)) = \underbrace{(E_1 \circ E_1 \circ E_1 \circ \ldots \circ E_1)}_{p+1 \text{ times}}(x)$$

belong also to the class $\mathcal{C}^\infty$ and we have

$$E_1(\mathbb{R}) = (0,\infty),\ E_2(\mathbb{R}) = E_1(0,\infty) = (1,\infty),$$
$$E_3(\mathbb{R}) = E_1(1,\infty) = (e,\infty),\ E_4(\mathbb{R}) = E_1(e,\infty) = (e^e,\infty),$$
$$E_5(\mathbb{R}) = E_1(e^e,\infty) = (e^{e^e},\infty)\ldots$$

Let us denote by $A_1, A_2, \ldots, A_p, \ldots$ the elements of $\mathbb{R}_+$ given by $A_1 = 0$, $A_2 = 1 = E_1(A_1)$, $A_3 = e = E_1(A_2)$, $A_4 = e^e = E_1(A_3) \ldots A_{p+1} = E_1(A_p) \ldots$.

Obviously we have $0 = A_1 < A_2 < A_3 < A_4 < \ldots < A_p < A_{p+1} < \ldots$ and the functions $E_1, E_2, \ldots, E_p \ldots$ defined on the interval $(-\infty,\infty)$ belong to the class $\mathcal{C}^\infty(\mathbb{R})$, they are strictly increasing and $E_1(-\infty,\infty) = (0,\infty) = (A_1,\infty)$, $E_2(-\infty,\infty) = (1,\infty) = (A_2,\infty)$, $E_3(-\infty,\infty) = (A_3,\infty)$, $\ldots, E_p(-\infty,\infty) = (A_p,\infty)$.

One may show inductively that

$$E_p'(x) = E_1(x)E_2(x)\cdots E_p(x), \quad \forall\, p \geq 1$$

and therefore, if we denote by l_p the inverse of the function E_p, $l_p : (A_p, \infty) \to (-\infty, \infty)$, the function l_p belongs to the class $\mathcal{C}^\infty(A_p, \infty)$ and we have

$$E_q \circ E_p = E_p \circ E_q = E_{p+q}, \quad E_p^{-1} \circ E_q^{-1} = E_q^{-1} \circ E_p^{-1} = E_{p+q}^{-1}.$$

Hence, for any $t \in E_{p+q}(-\infty, \infty) = (A_{p+q}, \infty)$ we have

$$E_{p+q}^{-1}(t) = E_p^{-1} \circ E_q^{-1}(t) = E_q^{-1} \circ E_p^{-1}(t);$$
$$l_{p+q}(t) = (l_p \circ l_q)(t) = (l_q \circ l_p)(t);$$
$$E_q(l_{q+p}(t)) = E_q(l_q(l_p(t))) = l_p(t)$$

and therefore

$$t \in (A_{p+q}, \infty) \Rightarrow l'_{p+q}(t) = \frac{1}{E'_{p+q}(l_{p+q}(t))} = \frac{1}{(E_1 E_2 \cdots E_{p+q})(l_{p+q}(t))} =$$
$$= \frac{1}{E_1(l_{p+q}(t)) E_2(l_{p+q}(t)) \cdot \ldots \cdot E_{p+q}(l_{p+q}(t))} = \frac{1}{l_{p+q-1}(t) \cdot l_{p+q-2}(t) \cdot \ldots \cdot l_1(t) \cdot t};$$
$$l'_n(t) = \frac{1}{t \cdot l_1(t) \cdot l_2(t) \cdot \ldots \cdot l_{n-1}(t)}, \quad \forall\, t \in (A_n, \infty).$$

We remark also that for any $p \geq 1$, $p \in \mathbb{N}$ we have

$$l_p(A_{p+1}, \infty) = (0, \infty) \quad \text{and} \quad l_p(A_{p+2}, \infty) = (1, \infty).$$

We shall denote by Δ_k, $k \in \mathbb{N}^*$, the function defined on (A_k, ∞) given by

$$\Delta_k(x) = l_k(x+1) - l_k(x).$$

Lemma 2. *a) For any $x \in [e, \infty) = [A_3, \infty)$ and any $y \geq x$ we have*

$$\frac{l_1(y)}{l_1(x)} \leq \frac{y}{x}.$$

b) For any $x \in [A_{k+2}, \infty)$ and any $y \geq x$ we have

$$\frac{l_k(y)}{l_k(x)} \leq \frac{y}{x}.$$

Proof. a) If we denote $u = l_1(x)$, $v = l_1(y)$ we have $1 \leq u \leq v$ and therefore

$$\frac{l_1(y)}{l_1(x)} = \frac{v}{u} = 1 + \frac{v-u}{u} \leq 1 + (v-u) \leq e^{v-u} = \frac{e^v}{e^u} = \frac{y}{x}.$$

b) The inequality may be done inductively. For $k = 1$ the assertion b) is just the assertion a). We suppose that for $k \geq 1$, $x \in [A_{k+2}, \infty)$, $y \geq x$ we have

$$\frac{l_k(y)}{l_k(x)} \leq \frac{y}{x}.$$

If $x \in [A_{k+3}, \infty)$ and $y \geq x$ we have $l_1(x) \in [A_{k+2}, \infty) \subset [A_3, \infty)$, $l_1(y) \geq l_1(x)$ and therefore by the hypothesis we have

$$\frac{l_k(l_1(y))}{l_k(l_1(x))} \leq \frac{l_1(y)}{l_1(x)} \leq \frac{y}{x}.$$

□

Lemma 3. *For any $k \in \mathbb{N}^*$ and any $x \in [A_{k+1}, \infty)$ we have*

$$0 \leq \frac{(x+1) l_1(x+1) l_2(x+1) \cdot \ldots \cdot l_{k-1}(x+1) \Delta_{k-1}(x)}{l_{k-1}(x)} - 1 \leq \frac{1}{x} \cdot 2^k.$$

Proof. Taking $k \in \mathbb{N}^*$, $x \geq A_{k+1}$ and applying Lagrange Theorem we deduce the existence of a real number $x' \in (x, x+1)$ such that

$$\Delta_{k-1}(x) = l_{k-1}(x+1) - l_{k-1}(x) = \frac{1}{x' l_1(x') l_2(x') \cdot \ldots \cdot l_{k-2}(x')}.$$

If we denote

$$F_{k-1}(x) = \frac{(x+1) l_1(x+1) l_2(x+1) \cdot \ldots \cdot l_{k-1}(x+1) \Delta_{k-1}(x)}{l_{k-1}(x)} - 1$$

we have

$$F_{k-1}(x) = \frac{x+1}{x'} \cdot \frac{l_1(x+1)}{l_1(x')} \cdot \frac{l_2(x+1)}{l_2(x')} \cdot \ldots \cdot \frac{l_{k-2}(x+1)}{l_{k-2}(x')} \cdot \frac{l_{k-1}(x+1)}{l_{k-1}(x)} - 1$$

Since the functions $l_1, l_2, \ldots, l_{k-1}$ are positive and increasing on the interval $[A_{k+1}, \infty)$ we deduce that the function F_{k-1} is positive on the interval $[A_{k+1}, \infty)$ and moreover we have

$$0 \leq F_{k-1}(x) \leq \frac{x+1}{x} \cdot \frac{l_1(x+1)}{l_1(x)} \cdot \frac{l_2(x+1)}{l_2(x)} \cdot \ldots \cdot \frac{l_{k-1}(x+1)}{l_{k-1}(x)} - 1.$$

We apply now Lemma 2 and we obtain

$$\frac{x+1}{x} = 1 + \frac{1}{x}, \ \frac{l_1(x+1)}{l_1(x)} \leq 1 + \frac{1}{x}, \ldots, \frac{l_{k-1}(x+1)}{l_{k-1}(x)} \leq 1 + \frac{1}{x},$$

$$F_{k-1}(x) \leq \left(1 + \frac{1}{x}\right)^k - 1 = \sum_{j=1}^{k} C_k^j \cdot \left(\frac{1}{x}\right)^j \leq \frac{1}{x} \sum_{j=1}^{k} C_k^j < \frac{1}{x} \cdot 2^k.$$ □

We remember now, under a convenient form, the well known Raabe-Duhamel and Gauss criterions of convergence (or divergence) for the series with positive terms (see [1] or [2]).

From now on $\sum a_n$ will be a series of real numbers such that $a_n > 0$ for all $n \in \mathbb{N}$.

Raabe-Duhamel divergence criterion. *If* $\frac{a_n}{a_{n+1}} \leq 1 + \frac{1}{n}$, *for n sufficiently large, then the series $\sum a_n$ is divergent.*

Raabe-Duhamel convergence criterion. *If $\alpha \in \mathbb{R}$, $\alpha > 1$ and for n sufficiently large we have* $\frac{a_n}{a_{n+1}} \geq 1 + \frac{\alpha}{n}$ *then the series $\sum a_n$ is convergent.*

Gauss divergence criterion. *If there exist $\alpha \in (1, \infty)$ and a (positive) real number M such that* $\frac{a_n}{a_{n+1}} \leq 1 + \frac{1}{n} + \frac{M}{n^\alpha}$, *for n sufficiently large, then the series $\sum a_n$ is divergent.*

Gauss convergence criterion. *If there exist $r \in (1, \infty)$, $\alpha \in (1, \infty)$ and M a (negative) real number such that* $\frac{a_n}{a_{n+1}} \geq 1 + \frac{r}{n} + \frac{M}{n^\alpha}$ *for n sufficiently large, then the series $\sum a_n$ is convergent.*

Kummer divergence criterion. *If (k_n) is a sequence of real numbers, $k_n > 0$, for all $n \in \mathbb{N}$ such that the series* $\sum_n \frac{1}{k_n}$ *is divergent and we have*

$$k_n \cdot \frac{a_n}{a_{n+1}} - k_{n+1} \leq 0$$

for n sufficiently large, then the series $\sum_n a_n$ *is divergent.*

Kummer convergence criterion. *If $(k_n)_n$ is a sequence of real numbers $k_n > 0$, for all $n \in \mathbb{N}$ and if there exists $\alpha > 0$ such that*

$$k_n \cdot \frac{a_n}{a_{n+1}} - k_{n+1} \geq \alpha$$

for n sufficiently large, then the series $\sum a_n$ is convergent.

Remark 1. If instead of the above sequence $(k_n)_n$ of positive numbers we take $k_n = n \ln n$ or $k_n = n \log_a n$, where $a \in (1, \infty)$, we obtain so called *Bertrand criterion.*

Remark 2. Even *Gauss criterion* is a consequence of Bertrand criterion.

Remark 3. The sequence $(k_n)_{n \geq A_{p+1}}$ of positive real numbers given by

$$k_n = n l_1(n) l_2(n) \cdot \ldots \cdot l_p(n),\ n \geq A_{p+1}$$

is increasing and the series $\sum\limits_{n \geq A_{p+1}} \frac{1}{k_n}$ is divergent. Here $p \in \mathbb{N}$ is arbitrary, $p \geq 1$.

2. The main result

From now on we shall use the notations from the preceding section, $\sum a_n$ will be a series of real number, $a_n > 0$ for all n and p will be a natural number, $p \geq 1$.

Theorem DT_p (p - divergence criterion). *If we have*

$$\frac{a_n}{a_{n+1}} \leq 1 + \frac{1}{n} + \frac{1}{n l_1(n)} + \frac{1}{n l_1(n) l_2(n)} + \ldots + \frac{1}{n l_1(n) l_2(n) \cdot \ldots \cdot l_p(n)},$$

for n sufficiently large, then the series $\sum a_n$ is divergent.

Proof. Let $k_n = n l_1(n) l_2(n) \cdot \ldots \cdot l_p(n)$ for $n \in \mathbb{N}$, $n \geq A_{p+1}$. We know that the series $\sum\limits_n \frac{1}{k_n}$ is divergent. We try to use Kummer divergence criterion. We have

$$k_n \cdot \frac{a_n}{a_{n+1}} - k_{n+1} \leq (n+1)(l_1 l_2 \cdot \ldots \cdot l_p)(n) + (l_2 l_3 \cdot \ldots \cdot l_p)(n) + (l_3 l_4 \cdot \ldots \cdot l_p)(n) + \ldots$$

$$\ldots + (l_{p-1} l_p)(n) + l_p(n) + 1 - (n+1)(l_1 l_2 \cdot \ldots \cdot l_p)(n+1) =$$
$$= [(n+1)(l_1(n) - l_1(n+1)) + 1] \cdot (l_2 l_3 \cdot \ldots \cdot l_p)(n) +$$
$$+ [(n+1) l_1(n+1)(l_2(n) - l_2(n+1)) + 1] \cdot (l_3 l_4 \cdot \ldots \cdot l_p)(n) +$$
$$\vdots$$
$$+ [(n+1) l_1(n+1) l_2(n+1) \cdot \ldots \cdot l_{s-1}(n+1)(l_s(n) - l_s(n+1)) + 1](l_{s+1} l_{s+2} \cdot \ldots \cdot l_p)(n) +$$
$$+ \ldots + [(n+1)(l_1 l_2 \cdot \ldots \cdot l_{p-2})(n+1)(l_{p-1}(n) - l_{p-1}(n+1)) + 1] \cdot l_p(n) +$$
$$+ [(n+1)(l_1 l_2 \cdot \ldots \cdot l_{p-1})(n+1)(l_p(n) - l_p(n+1)) + 1].$$

Obviously $l_s(n) > 0$ for all $s \leq p$ and any $n \geq A_{p+1}$. To finish the proof it will be sufficient to show that

$$(n+1)(l_1 l_2 \cdot \ldots \cdot l_{s-1})(n+1)(l_s(n) - l_s(n+1)) + 1 \leq 0,\ \forall\, n \geq A_{p+1}.$$

This inequality follows applying Lagrange Theorem, namely there exists x, $n < x < n+1$ such that

$$l_s(n+1) - l_s(n) = \frac{1}{x \cdot (l_1 l_2 \cdot \ldots \cdot l_{s-1})(x)}$$

and therefore

$$(n+1)(l_1 l_2 \cdot \ldots \cdot l_{s-1})(n+1) \cdot (l_s(n) - l_s(n+1)) + 1 =$$
$$= -\left(\frac{n+1}{x}\right) \cdot \frac{l_1(n+1)}{l_1(x)} \cdot \frac{l_2(n+1)}{l_2(x)} \cdot \ldots \cdot \frac{l_{s-1}(n+1)}{l_{s-1}(x)} + 1 < 0.$$

□

Theorem CT_p (p - convergence criterion). *If there exists* $\alpha > 1$ *such that*

$$\frac{a_n}{a_{n+1}} \geq 1 + \frac{1}{n} + \frac{1}{nl_1(n)} + \ldots + \frac{1}{nl_1(n)l_2(n) \cdot \ldots \cdot l_{p-1}(n)} + \frac{\alpha}{nl_1(n)l_2(n) \cdot \ldots \cdot l_p(n)},$$

for n *sufficiently large, then the series* $\sum a_n$ *is convergent.*

Proof. Using the same sequence $(k_n)_n$ we shall use again Kummer (convergence) criterion. We shall try to show that for any $n \in \mathbb{N}$, sufficiently large, we have

$$k_n \cdot \frac{a_n}{a_{n+1}} - k_{n+1} \geq r > 0, \quad \text{where } r = \frac{\alpha - 1}{2}.$$

From the calculus performed in the proof of Theorem DT_p we have

$$k_n \cdot \frac{a_n}{a_{n+1}} - k_{n+1} \geq [(n+1)(l_1(n) - l_1(n+1)) + 1](l_2 l_3 \cdot \ldots \cdot l_p)(n) +$$
$$+ \sum_{s=2}^{p-1} [(n+1)(l_1 l_2 \cdot \ldots \cdot l_{s-1})(n+1)(l_s(n) - l_s(n+1)) + 1] \cdot (l_{s+1} l_{s+2} \cdot \ldots \cdot l_p)(n) +$$
$$+ [(n+1)(l_1 l_2 \cdot \ldots \cdot l_{p-1})(n+1)(l_p(n) - l_p(n+1)) + 1] + 2r.$$

To finish the proof it will be sufficient to show that

$$\lim_{n\to\infty} [(n+1)(l_1 l_2 \cdot \ldots \cdot l_{s-1})(n+1)(l_s(n) - l_s(n+1)) + 1] \cdot (l_{s+1} l_{s+2} \cdot \ldots \cdot l_p)(n) = 0$$

for $s \in \{2, 3, \ldots, p-1\}$ and

$$\lim_{n\to\infty} [(n+1)(l_1(n) - l_1(n+1)) + 1](l_2 l_3 \cdot \ldots \cdot l_p)(n) = 0 =$$
$$= \lim_{n\to\infty} [(n+1)(l_1 \cdot \ldots \cdot l_{p-1})(n+1)(l_p(n) - l_p(n+1)) + 1].$$

For our purpose we shall use the above Lemma 1 and Lemma 3. We have

$$0 \leq (n+1)(l_1 l_2 \cdot \ldots \cdot l_{s-1})(n+1)(l_s(n+1) - l_s(n)) - 1 =$$
$$= (n+1)(l_1 l_2 \cdot \ldots \cdot l_{s-1})(n+1) \cdot \left(l_1 \left(\frac{l_{s-1}(n+1)}{l_{s-1}(n)} \right) \right) - 1 =$$
$$= \frac{(n+1)(l_1 l_2 \cdot \ldots \cdot l_{s-1})(n+1)\Delta_{s-1}(n)}{l_{s-1}(n)} \cdot l_1 \left[\left(1 + \frac{\Delta_{s-1}(n)}{l_{s-1}(n)} \right)^{\frac{l_{s-1}(n)}{\Delta_{s-1}(n)}} \right] - 1 \leq$$
$$\leq \left(\frac{(n+1)(l_1 l_2 \cdot \ldots \cdot l_{s-1})(n+1)\Delta_{s-1}(n)}{l_{s-1}(n)} - 1 \right) l_1 \left[\left(1 + \frac{\Delta_{s-1}(n)}{l_{s-1}(n)} \right)^{\frac{l_{s-1}(n)}{\Delta_{s-1}(n)}} \right] \leq$$
$$\leq \frac{1}{n} \cdot 2^s$$

and therefore, if $n \geq A_{p+1}$, the following inequality holds

$$0 \leq [(n+1)(l_1 l_2 \cdot \ldots \cdot l_{s-1})(n+1)(l_s(n+1) - l_s(n)) - 1](l_{s+1} \cdot \ldots \cdot l_p)(n) \leq$$
$$\leq 2^s \cdot \frac{(l_{s+1} \cdot \ldots \cdot l_p)(n)}{n}.$$

Hence

$$\lim_{n\to\infty} [(n+1)(l_1 l_2 \cdot \ldots \cdot l_{s-1})(n+1)(l_s(n+1) - l_s(n)) - 1](l_{s+1} \cdot \ldots \cdot l_p)(n) = 0.$$

In a similar way one can show the assertions

$$0 = \lim_{n\to\infty} [(n+1)(l_1(n) - l_1(n+1)) + 1](l_2 l_3 \cdot \ldots \cdot l_p)(n),$$
$$\lim_{n\to\infty} [(n+1)(l_1 l_2 \cdot \ldots \cdot l_{p-1})(n+1)(l_p(n) - l_p(n+1)) - 1] = 0.$$

Hence $k_n \cdot \dfrac{a_n}{a_{n+1}} - k_{n+1} > r$ for n sufficiently large. □

Theorem LT_p **.** *If there exists the following limit*

$$l := \lim_{n\to\infty} n(l_1 l_2 \cdot \ldots \cdot l_p)(n)\Big(\frac{a_n}{a_{n+1}} - 1 - \frac{1}{n} - \frac{1}{nl_1(n)} - \frac{1}{n(l_1 l_2)(n)} - \ldots$$
$$\ldots - \frac{1}{n(l_1 l_2 \cdot \ldots \cdot l_{p-1})(n)}\Big)$$

then the series $\sum_n a_n$ *converges for* $l > 1$ *and diverges for* $l < 1$.

3. Examples and counterexamples

In this part we establish some relations between different convergence (or divergence) criterions for series with positive terms.

Notations. If C', C'' are two criterions of convergence (or divergence) for series $\sum a_n$, with $a_n > 0$ we say that C'' *is stronger than* C' and we write $C' \le C''$, if the conditions in which C'' acts are automatically fulfilled whenever the conditions in which C' acts are fulfilled.

Proposition. *If we denote by CR, CG (respectively DR, DG) the Raabe convergence, Gauss convergence (respectively Raabe divergence, Gauss divergence) criterions then we have*

$$CR < CG < CT_1 < CT_2 < CT_3 < \ldots < CT_p < CT_{p+1} < \ldots$$
$$DR < DG < DT_1 < DT_2 < DT_3 < \ldots < DT_p < DT_{p+1} < \ldots$$

Proof. From the inequalities

$$1 + \frac{1}{n} \le 1 + \frac{1}{n} + \frac{M}{n^\alpha} \le 1 + \frac{1}{n} + \frac{1}{nl_1(n)} \le 1 + \frac{1}{n} + \frac{1}{nl_1(n)} + \frac{1}{n(l_1 l_2)(n)} \le \ldots$$
$$\ldots \le 1 + \frac{1}{n} + \frac{1}{nl_1(n)} + \frac{1}{n(l_1 l_2)(n)} + \frac{1}{n(l_1 l_2 l_3)(n)} + \ldots + \frac{1}{n(l_1 l_2 \cdot \ldots \cdot l_p)(n)}$$

for any $\alpha > 1$, $M > 0$ and n sufficiently large, $n \ge A_{p+1}$, we deduce that

$$DR \le DG \le DT_1 \le DT_2 \le \ldots \le DT_p.$$

Some examples will show that these inequalities are strict.

If $b_1, b_2, \ldots, b_k$ are real numbers we shall denote by $\prod_{i\le k} b_i$ their product $b_1 \cdot b_2 \cdot \ldots \cdot b_k$.

Let us take the sequence $(a_n)_n$ in $\mathbb{R}_+$ given by

$$a_n = \frac{((n-1)!)^2}{\prod_{k\le n-1}(1 + k + k^2)}.$$

We have

$$\frac{a_n}{a_{n+1}} = \frac{n^2 + n + 1}{n^2} = 1 + \frac{1}{n} + \frac{1}{n^2}.$$

DG criterion decides the divergence of the series $\sum a_n$ but DR criterion doesn't it. Hence $DR < DG$.

Let now $b_n \in \mathbb{R}_+$ given by

$$b_n = \frac{(n-1)!(\ln(2))(\ln(3)) \cdot \ldots \cdot (\ln(n-1))}{(1 + 3\ln 2)(1 + 4\ln 3) \cdot \ldots \cdot (1 + n\ln(n-1))},\ n \ge 3.$$

We have

$$\frac{b_n}{b_{n+1}} = \frac{1 + (n+1)\ln(n)}{n\ln(n)} = 1 + \frac{1}{n} + \frac{1}{n\ln(n)}.$$

By DT_1 - criterion the series $\sum\limits_{n\geq 3} b_n$ is divergent. In the same time if we take $\alpha > 1$ and $M > 0$ we can not have the inequalities

$$\frac{b_n}{b_{n+1}} \leq 1 + \frac{1}{n} + \frac{M}{n^\alpha}$$

at least for a sufficiently large n because the inequality $\dfrac{1}{n\ln(n)} \leq \dfrac{M}{n^\alpha}$ fails for n sufficiently large.

Hence DG - criterion does not decide the nature of the series $\sum b_n$ i.e. $DG < DT_1$.

Let now $p \in \mathbb{N}$, $p \geq 2$ and $n_0 \in \mathbb{N}$ be the smallest natural number greater than A_{p+1}, defined in Section 1. We consider a sequence $(b_n)_{n\geq n_0}$ inductively defined by

$$b_{n_0} = 1,$$

$$b_{n+1} = b_n \cdot \frac{n(l_1 l_2 \cdot \ldots \cdot l_p)(n)}{1 + (n+1)(l_1 l_2 \cdot \ldots \cdot l_p)(n) + (l_2 l_3 \cdot \ldots \cdot l_p)(n) + (l_3 l_4 \cdot \ldots \cdot l_p)(n) + \ldots + l_p(n)}.$$

Obviously $b_n > 0$ and

$$\frac{b_n}{b_{n+1}} = 1 + \frac{1}{n} + \frac{1}{nl_1(n)} + \ldots + \frac{1}{n(l_1 l_2 \cdot \ldots \cdot l_{p-1})(n)} + \frac{1}{n(l_1 l_2 \cdot \ldots \cdot l_p)(n)}.$$

Using DT_p - criterion we decide that the series $\sum\limits_{n\geq n_0} b_n$ is divergent. Since

$$1 + \frac{1}{n} + \frac{1}{nl_1(n)} + \ldots + \frac{1}{n(l_1 l_2 \cdot \ldots \cdot l_{p-1})(n)} < \frac{b_n}{b_{n+1}},\ \forall\, n \geq n_0,$$

the DT_{p-1} - criterion does not decide the nature of the series $\sum\limits_{n\geq n_0} b_n$, i.e. $DT_{p-1} < DT_p$. From the preceding considerations we get $DR < DG < DT_1 < DT_2 < \ldots < DT_p$.

We show now that $CR < CG < CT_1 < CT_2 < \ldots < CT_p$.

The relations $CR \leq CG \leq CT_1$ are obvious.

For an arbitrary $p \in \mathbb{N}$, $p \geq 1$ we consider a series $\sum a_n$ for which there exists $\alpha > 1$ such that

$$\frac{a_n}{a_{n+1}} \geq 1 + \frac{1}{n} + \frac{1}{nl_1(n)} + \frac{1}{n(l_1 l_2)(n)} + \ldots + \frac{1}{n(l_1 l_2 \cdot \ldots \cdot l_{p-1})(n)} + \frac{\alpha}{n(l_1 l_2 \cdot \ldots \cdot l_p)(n)}.$$

Since

$$\lim n(l_1 l_2 \cdot \ldots \cdot l_{p+1})\left(\frac{a_n}{a_{n+1}} - 1 - \frac{1}{n} - \frac{1}{nl_1(n)} - \ldots - \frac{1}{n(l_1 \cdot \ldots \cdot l_p)(n)}\right) \geq$$
$$\geq (\alpha - 1)\infty > 2,$$

we have for n sufficiently large

$$\frac{a_n}{a_{n+1}} \geq 1 + \frac{1}{n} + \frac{1}{nl_1(n)} + \ldots + \frac{1}{n(l_1 \cdot \ldots \cdot l_p)(n)} + \frac{2}{n(l_1 \cdot \ldots \cdot l_{p+1})(n)}.$$

Hence $CT_p \leq CT_{p+1}$, $CR \leq CG \leq CT_1 \leq CT_2 \leq \ldots \leq CT_p$.

The fact that we have strict inequalities before, may be shown by some examples. More precisely we consider $p \in \mathbb{N}$, $p > 1$ and for any $n \geq [A_{p+1}] + 1 = n_0$ let $b_n \in \mathbb{R}_+$ given by

$$b_{n_0} = 1,$$

$$b_{n+1} = b_n \cdot \frac{n(l_1 l_2 \cdot \ldots \cdot l_p)(n)}{2 + l_p(n) + (l_{p-1} l_p)(n) + \ldots + (l_2 l_3 \cdot \ldots \cdot l_p)(n) + (n+1)(l_1 l_2 \cdot \ldots \cdot l_p)(n)}.$$

We have

$$\frac{b_n}{b_{n+1}} = 1 + \frac{1}{n} + \frac{1}{nl_1(n)} + \frac{1}{n(l_1 l_2)(n)} + \ldots + \frac{1}{n(l_1 l_2 \cdot \ldots \cdot l_{p-1})(n)} + \frac{2}{n(l_1 \cdot \ldots \cdot l_p)(n)}$$

and therefore using CT_p - criterion the series $\sum b_n$ is convergent. In the same time there is no $\alpha > 1$ such that

$$\frac{b_n}{b_{n+1}} \geq 1 + \frac{1}{n} + \frac{1}{nl_1(n)} + \frac{1}{n(l_1 l_2)(n)} + \ldots + \frac{1}{n(l_1 l_2 \cdot \ldots \cdot l_{p-2})(n)} + \frac{\alpha}{n(l_1 l_2 \cdot \ldots \cdot l_{p-1})(n)}$$

at least for n sufficiently large. Hence CT_{p-1} - criterion does not decide the convergence of the series $\sum b_n$ i.e. $CT_{p-1} < CT_p$.

References

[1] N. Boboc, *Mathematical Analysis* (in Romanian), vol. 1, Editura Universităţii Bucureşti, 1999.

[2] M. Nicolescu, *Mathematical Analysis* (in Romanian), vol. 1, Editura Tehnică, 1957.

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