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COFINITENESS AND ARTINIANNESS OF GENERALIZED LOCAL COHOMOLOGY MODULES

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Abstract. Let R be a commutative Noetherian ring, $\mathfrak{a}$ and $\mathfrak{b}$ ideals of R and let M and N be two finitely generated R-modules. In this paper, we study the cofiniteness of $H^j_{\mathfrak{b}}(H^i_{\mathfrak{a}}(M,N))$ in several cases.

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1. Introduction

Throughout this paper, R will denote a commutative Noetherian (not necessarily local) ring, and M, N are two finitely generated R-modules. Also, $\mathfrak{a}$ and $\mathfrak{b}$ will denote two proper ideals of R.

Let $H^i_{\mathfrak{a}}(M,N) = \varinjlim_{n\in\mathbb{N}} \operatorname{Ext}^i_R(M/\mathfrak{a}^n M, N)$ be the i-th generalized local cohomology module relative to the ideal $\mathfrak{a}$ and R-modules M and N (see [8]). For $M = R$, let us denote $H^i_{\mathfrak{a}}(R,N)$ by $H^i_{\mathfrak{a}}(N)$, the i-th ordinary local cohomology module with respect to $\mathfrak{a}$. In [6] Grothendieck conjectured that for any ideal $\mathfrak{a}$ and for any finite generated R-module N, the R-module $\operatorname{Hom}(R/\mathfrak{a}, H^i_{\mathfrak{a}}(N))$ is finite generated. In an Inventiones Mathematicae paper (see [7]) Hartshorn gives a counterexample to this conjecture and makes some additional assumptions to the original proposal of Grothendieck, introducing for instance the notion of $\mathfrak{a}$-cofiniteness for a module. He defined an R-module T to be $\mathfrak{a}$-cofinite if $\operatorname{Ext}^i_R(R/\mathfrak{a}, T)$ is finitely generated for all $i \geq 0$ and $\operatorname{Supp}T \subseteq V(\mathfrak{a})$, where $V(\mathfrak{a})$ denotes the set of prime ideals of R containing $\mathfrak{a}$, and asked the following question:

Let N be a finitely generated R-module and let $\mathfrak{a}$ be an ideal of R. Then, is $H^i_{\mathfrak{a}}(N)$ $\mathfrak{a}$-cofinite ?

This question has been studied by several authors; see for example, Yoshida [15], Zamani [14], Cuong, Goto and Hong [12], Dehghani-Zadeh [3], Bahmanpour and Naghipour [2].

In this note the following question is of interest: Are the modules $H^j_{\mathfrak{b}}(H^i_{\mathfrak{a}}(M,N))$, $\mathfrak{b}$-cofinite?
The main purpose of this paper is to provide an affirmative answer to this question. In this direction as the result of this paper we prove $H^j_{\mathfrak{b}}(H^i_{\mathfrak{a}}(M,N))$ is $\mathfrak{b}$-cofinite, in the following cases:

(i) $\dim R/\mathfrak{a} \leq 1$ and $\dim R/\mathfrak{b} \leq 1$.
(ii) $\dim R/\mathfrak{a} = 2$ and $\dim R/\mathfrak{b} = 1$ and $i \leq f_{\mathfrak{a}}(M,N)$, where $f_{\mathfrak{a}}(M,N)$ is the least non-negative integer i such that $H^i_{\mathfrak{a}}(M,N)$ is not finitely generated.

In addition, we assume that R is a local ring with its maximal ideal $\mathfrak{m}$ and we study in what conditions on "i" the module $H^j_{\mathfrak{b}}(H^i_{\mathfrak{a}}(M,N))$ is $\mathfrak{b}$-cofinite, does not matter the number $\dim R/\mathfrak{b}$ and $\dim R/\mathfrak{a}$ are.

2. Cofiniteness of $H^j_{\mathfrak{b}}(H^i_{\mathfrak{a}}(M,N))$ for ideals of small dimension.

The concept of a cofinite module plays an important role in this paper. We say that T is a cofinite module if there is a proper ideal I of R such that T is I-cofinite. In this section, we study the cofiniteness of the modules $H^j_{\mathfrak{b}}(H^i_{\mathfrak{a}}(M,N))(i,j\in\mathbb{N}_0)$, where $\mathfrak{a},\mathfrak{b}$ are ideals in an arbitrary Noetherian (not necessarily local) ring R with $\mathfrak{a}\subseteq\mathfrak{b}$ and M, N finitely generated modules over R.

For any unexplained notation and terminology, we refer the reader to [1] and [13].

The following remark, which is needed in the proof of the next theorems, describes some of properties of cofinite modules.

Remark 2.1.
- (i) Assume that T is an $\mathfrak{a}$-cofinite R-module. Then T is $\mathfrak{a}$-torsion-free if and only if $\mathfrak{a}$ contains an element x which is T-regular. (see a proof in [1, Lemma 2.1.1] for instance).
- (ii) The class of Artinian $\mathfrak{a}$-cofinite modules is closed under taking submodules, quotients and extensions. (see [10, Corollary 4.4]).
- (iii) Let T and T' be two $\mathfrak{a}$-cofinite modules. If $f: T\to T'$ is a homomorphism between these two $\mathfrak{a}$-cofinite modules and one of the three modules $\mathrm{Ker}f$, $\mathrm{Im}f$ and $\mathrm{Coker}f$ is $\mathfrak{a}$-cofinite, then all three of them are $\mathfrak{a}$-cofinite.
- (iv) If R is a local ring with its maximal ideal $\mathfrak{m}$, then an R-module is $\mathfrak{m}$-cofinite if and only if it is an Artinian R-module (see [9]).
- (v) For each R-module T, set $\Gamma_{\mathfrak{b}}(T)=\bigcup_{n\in\mathbb{N}}(0:_T\mathfrak{b}^n)$, the set of elements of T which are annihilated by some power of $\mathfrak{b}$.

Theorem 2.2. *Let $\mathfrak{a}\subseteq\mathfrak{b}$ be two ideals of R such that $\dim R/\mathfrak{b}=0$. Let T be an $\mathfrak{a}$-cofinite R-module and M be a finitely generated R-module. Then $H^i_{\mathfrak{b}}(M,T)$ is an Artinian, $\mathfrak{a}$ and $\mathfrak{b}$-cofinite R-module.*

Proof. Firstly, we provide some facts which are needed in the course of the proof. As T is $\mathfrak{a}$-cofinite, the R-module $\mathrm{Hom}(R/\mathfrak{a},T)$ is finitely generated. Hence $\mathrm{Hom}(R/\mathfrak{b},T)$ and $\mathrm{Hom}(R/\mathfrak{b},\Gamma_{\mathfrak{b}}(T))$ are finitely generated R-modules. Since $\dim R/\mathfrak{b}=0$, it follows that $\mathrm{Hom}(R/\mathfrak{b},\Gamma_{\mathfrak{b}}(T))$ is of finite length. Therefore, by [10, Proposion 4.1], we deduce that $\Gamma_{\mathfrak{b}}(T)$ is an $\mathfrak{b}$-cofinite and Artinian R-module. In addition, finiteness of $\mathrm{Hom}(R/\mathfrak{a},T)$ shows that, $\mathrm{Hom}\big(R/\mathfrak{a},\Gamma_{\mathfrak{b}}(T)\big)$ is finitely generated. According to Melkersson [10, Proposition 4.1], $\Gamma_{\mathfrak{b}}(T)$ is an Artinian and $\mathfrak{a}$-cofinite R-module. Now we use mathematical induction on "i". If $i=0$, then $H^0_{\mathfrak{b}}(M,N)\cong\mathrm{Hom}(M,\Gamma_{\mathfrak{b}}(T))$, and the assertion is trivial, by Remark (2.1, ii). Let $i>0$ and we assume that the result is true for $i-1$. Let us consider the exact sequence

$$H^i_{\mathfrak{b}}(M,\Gamma_{\mathfrak{b}}(T))\longrightarrow H^i_{\mathfrak{b}}(M,T)\longrightarrow H^i_{\mathfrak{b}}\big(M,T/\Gamma_{\mathfrak{b}}(T)\big),$$

in conjunction with the fact that $H^i_{\mathfrak{b}}\big(M,\Gamma_{\mathfrak{b}}(T)\big)\cong\mathrm{Ext}^i_R\big(M,\Gamma_{\mathfrak{b}}(T)\big)$, to see that $H^i_{\mathfrak{b}}(M,T)$ is Artinian and $\mathfrak{b}$-cofinite if and only if $H^i_{\mathfrak{b}}\big(M,T/\Gamma_{\mathfrak{b}}(T)\big)$ is Artinian and $\mathfrak{b}$-cofinite. We assume that $\Gamma_{\mathfrak{b}}(T)=0$. Then, in view of Remark (2.1, i), the ideal $\mathfrak{b}$ contains an element x which is T-regular. Now, let us look at the exact sequence $0\longrightarrow T\xrightarrow{x}T\longrightarrow T/xT\longrightarrow 0$ which gives rise to the exact sequence

$$H^{i-1}_{\mathfrak{b}}(M,T/xT)\longrightarrow H^i_{\mathfrak{b}}(M,T)\xrightarrow{x}H^i_{\mathfrak{b}}(M,T).$$

Now, the above exact sequence is used in conjunction with the inductive hypothesis and Remark (2.1, ii) to see that $\big(0:_{H^i_{\mathfrak{b}}(M,T)}x\big)$ is Artinian and $\mathfrak{b}$-cofinite. Hence, by [10, Proposition 4.1], $H^i_{\mathfrak{b}}(M,T)$ is Artinian and $\mathfrak{b}$-cofinite. In the same way we can prove that $H^i_{\mathfrak{b}}(M,T)$ is also $\mathfrak{a}$-cofinite. □

Corollary 2.3. *Let $\mathfrak{a}\subseteq\mathfrak{b}$ be two ideals of R such that $\dim R/\mathfrak{b}=0$ and $\dim R/\mathfrak{a}=1$. Then for each $j,i\geq 0$, $H^j_{\mathfrak{b}}(H^i_{\mathfrak{a}}(M,N))$ is an Artinian and $\mathfrak{b}$ and $\mathfrak{a}$-cofinite R-module.*

Proof. It follows from Theorem 2.2 and [3, Theorem 3.3]. □

Theorem 2.4. *Let $\mathfrak{a} \subseteq \mathfrak{b}$ be two ideals of R such that $\dim R/\mathfrak{b} = \dim R/\mathfrak{a} = 1$. If T is $\mathfrak{a}$-cofinite and i a positive integer, then $Ext_R^{i-1}\big(R/\mathfrak{b}, H_{\mathfrak{b}}^1(T)\big)$ is a finitely generated R-module if and only if $Ext_R^{i+1}\big(R/\mathfrak{b}, \Gamma_{\mathfrak{b}}(T)\big)$ is a finitely generated R-module.*

Proof. By [11, Theorem 11.38], there exists a Grothendieck's spectral sequence

$$E_2^{p,q} = \mathrm{Ext}_R^p\big(R/\mathfrak{b}, H_{\mathfrak{b}}^q(T)\big) \overset{p}{\Longrightarrow} \mathrm{Ext}_R^{p+q}\big(R/\mathfrak{b}, T\big). \qquad (*)$$

Since $\mathrm{Supp}T \subseteq V(\mathfrak{a})$ and $\dim R/\mathfrak{a} = 1$, it follows that $\dim(T) \leq 1$. This implies that R-module $H_{\mathfrak{b}}^q(T) = 0$ for $q > 1$ (see [1, Theorem 6.1.2]). Hence $E_2^{p,q} = 0$ unless $q = 0, 1$. Therefore, using the spectral sequence $(*)$ with [13, Exercise 5.2.2], the long exact sequence is resulted, which is following:

$$\mathrm{Ext}_R^{i+1}\big(R/\mathfrak{b}, T\big) \overset{\varphi}{\longrightarrow} \mathrm{Ext}_R^{i+1}\big(R/\mathfrak{b}, H_{\mathfrak{b}}^0(T)\big) \overset{d}{\longrightarrow} \mathrm{Ext}_R^{i-1}\big(R/\mathfrak{b}, H_{\mathfrak{b}}^1(T)\big)$$
$$\overset{\psi}{\longrightarrow} \mathrm{Ext}_R^{i}\big(R/\mathfrak{b}, T\big) \longrightarrow \mathrm{Ext}_R^{i}\big(R/\mathfrak{b}, H_{\mathfrak{b}}^0(T)\big).$$

In view of hypothesis and [4, Corollary 1], $\mathrm{Ext}_R^i(R/\mathfrak{b}, T)$ is finitely generated for all i. Hence $\mathrm{Im}\varphi$ and $\mathrm{Im}\psi$ are finitely generated. This proves the claim. □

Theorem 2.5. *Let $\mathfrak{a} \subseteq \mathfrak{b}$ be two ideals of R such that $\dim R/\mathfrak{b} = \dim R/\mathfrak{a} = 1$. Then $H_{\mathfrak{b}}^j\big(H_{\mathfrak{a}}^i(M,N)\big)$ is $\mathfrak{b}$-cofinite for all i and j.*

Proof. Consider the Grothendieck spectral sequence

$$E_2^{p,q} = H_{\mathfrak{b}}^p\big(H_{\mathfrak{a}}^q(M,N)\big) \overset{p}{\Longrightarrow} H_{\mathfrak{b}}^{p+q}(M,N). \qquad (**)$$

Since $\mathrm{Supp}H_{\mathfrak{a}}^q(M,N) \subseteq V(\mathfrak{a})$ and $\dim R/\mathfrak{a} = 1$, it follows that $E_2^{p,q} = 0$ unless $p = 0, 1$. Referring [13, Exercise 5.2.1], the spectral sequence $(**)$ results to the following short exact sequence:

$$0 \longrightarrow H_{\mathfrak{b}}^1\big(H_{\mathfrak{a}}^{i-1}(M,N)\big) \longrightarrow H_{\mathfrak{b}}^i(M,N) \longrightarrow H_{\mathfrak{b}}^0\big(H_{\mathfrak{a}}^i(M,N)\big) \longrightarrow 0.$$

Thus, there is a long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}_R^n\big(R/\mathfrak{b}, H_{\mathfrak{b}}^i(M,N)\big) \longrightarrow \mathrm{Ext}_R^n\big(R/\mathfrak{b}, H_{\mathfrak{b}}^0(H_{\mathfrak{a}}^i(M,N))\big) \longrightarrow$$
$$\longrightarrow \mathrm{Ext}_R^{n+1}\big(R/\mathfrak{b}, H_{\mathfrak{b}}^1(H_{\mathfrak{a}}^{i-1}(M,N))\big) \longrightarrow \mathrm{Ext}_R^{n+1}\big(R/\mathfrak{b}, H_{\mathfrak{b}}^i(M,N)\big) \longrightarrow \cdots . (\ddagger)$$

In view of [3, Theorem 3.3], $H_{\mathfrak{b}}^i(M,N)$ is $\mathfrak{b}$-cofinite and $H_{\mathfrak{a}}^i(M,N)$ is $\mathfrak{a}$-cofinite for all i. Therefore, using the exact sequence $(\ddagger)$ and Theorem 2.4 the result follows. □

Lemma 2.6. *Let $H_{\mathfrak{a}}^i(N)$ be Artinian for all $i < t$. Then $H_{\mathfrak{a}}^i(M,N)$ is Artinian and $\mathfrak{a}$-cofinite for all $i < t$.*

Proof. Since $H_{\mathfrak{a}}^i(N)$ is Artinian for all $i < t$, it follows that $\mathrm{Supp}H_{\mathfrak{a}}^i(N)$ is a finite set. Hence, by [2, Theorem 2.6], the R-module $H_{\mathfrak{a}}^i(N)$ is also $\mathfrak{a}$-cofinite. The assertion follows from [4, Theorem 2.1] and Remark (2.1,ii). □

The following Corollary is an immediate consequence of Lemma 2.6.

Corollary 2.7. *If $\dim R/\mathfrak{a} = 0$, then $H_{\mathfrak{a}}^i(M,N)$ is Artinian and $\mathfrak{a}$-cofinite for all i.*

Theorem 2.8. *Let $\mathfrak{a}$, $\mathfrak{b}$ be two ideals of R such that $\dim R/\mathfrak{b} = \dim R/\mathfrak{a} = 0$. Then $H_{\mathfrak{b}}^j(H_{\mathfrak{a}}^i(M,N))$ is $\mathfrak{b}$-cofinite for all i, j.*

Proof. Since, for each $i, j \geq 0$, $\mathrm{Supp}H_{\mathfrak{b}}^j(H_{\mathfrak{a}}^i(M,N)) \subseteq V(\mathfrak{b})$, it is enough to show that

$$Ext_R^t\big(R/\mathfrak{b}, H_{\mathfrak{b}}^j(H_{\mathfrak{a}}^i(M,N))\big)$$

is finitely generated for all $t \geq 0$. By using the previous corollary $H_{\mathfrak{a}}^i(M,N)$ is $\mathfrak{a}$-cofinite and Artinian, and so $H_{\mathfrak{b}}^0(H_{\mathfrak{a}}^i(M,N))$ is $\mathfrak{a}$-cofinite and Artinian. Since $\mathfrak{a} \subseteq \mathfrak{b}$, it follows from [5, Corollary1] that $Ext_R^t\big(R/\mathfrak{b}, H_{\mathfrak{b}}^0(H_{\mathfrak{a}}^i(M,N))\big)$ is finitely generated, for all $t \geq 0$. As $\mathrm{Supp}H_{\mathfrak{a}}^i(M,N) \subseteq V(\mathfrak{a})$ and $\dim R/\mathfrak{a} = 0$, it follows that $H_{\mathfrak{b}}^j(H_{\mathfrak{a}}^i(M,N)) = 0$ for all $j > 0$. This completes the proof. □

Definition 2.9. Let $\mathfrak{a}$ be a proper ideal of R. The number

$$f_{\mathfrak{a}}(M,N) = \inf\big\{i \in \mathbb{N}_0 \mid H^i_{\mathfrak{a}}(M,N) \text{ is not finitely generated}\big\},$$

is called the finiteness dimension of M and N relative to the ideal $\mathfrak{a}$.
The arithmetic rank of an ideal $\mathfrak{a}$, denoted by $\text{ara}(\mathfrak{a})$, is the least number of generates of all ideals $\mathfrak{c}$ which have the same radical as $\mathfrak{a}$.

Theorem 2.10. *Let $\mathfrak{a} \subseteq \mathfrak{b}$ be two proper ideals of R such that $\dim R/\mathfrak{a} = 2$ and $\dim R/\mathfrak{b} = 1$. Let $f_{\mathfrak{a}}(M,N) = f$. Then $H^0_{\mathfrak{b}}(H^f_{\mathfrak{a}}(M,N))$ is $\mathfrak{b}$-cofinite and for all $i < f$ and any j, $H^j_{\mathfrak{b}}\big(H^i_{\mathfrak{a}}(M,N)\big)$ is a $\mathfrak{b}$-cofinite R-module too.*

Proof. As $\dim R/\mathfrak{a} = 2$ and $\dim R/\mathfrak{b} = 1$, there is $x \in \mathfrak{b}$ such that $\dim R/(\mathfrak{a} + Rx) = 1$. This is by [11, Theorem 11.38], the Grothendieck spectral sequence $E_2^{p,q} = H^p_{xR}(H^q_{\mathfrak{a}}(M,N))$ converges to $H^{p+q} = H^{p+q}_{Rx+\mathfrak{a}}(M,N)$. As $\text{ara}(Rx) = 1$, it is easy to see that $E_2^{p,q} = 0$ unless $p = 0, 1$; it follows that the sequence $0 \to E_2^{1,f-1} \to H^f \to E_2^{0,f} \to 0$ is exact, which, in turn, yields the exact sequence

$$H^1_{xR}(H^{f-1}_{\mathfrak{a}}(M,N)) \to H^f_{Rx+\mathfrak{a}}(M,N) \to H^0_{Rx}(H^f_{\mathfrak{a}}(M,N)) \to 0. \qquad (\S)$$

In view of Definition 2.9, the R-module $H^{f-1}_{\mathfrak{a}}(M,N)$ is finitely generated. Therefore, by [10, Proposion 5.1] the R-module $H^1_{Rx}(H^{f-1}_{\mathfrak{a}}(M,N))$ is Rx-cofinite and Artinian. So, $Ext^t_R\big(R/Rx, H^1_{Rx}(H^{f-1}_{\mathfrak{a}}(M,N))\big)$ is a finitely generated R-module for all t. In view of [5, Corollary 1], $Ext^t_R\big(R/(Rx+\mathfrak{a}), H^1_{Rx}(H^{f-1}_{\mathfrak{a}}(M,N))\big)$ is a finitely generated R-module. Also, as $\text{Supp}H^1_{Rx}(H^{f-1}_{\mathfrak{a}}(M,N)) \subseteq V(Rx+\mathfrak{a})$ we get that $H^1_{Rx}(H^{f-1}_{\mathfrak{a}}(M,N))$ is Artinian and $(Rx+\mathfrak{a})$-cofinite. Now, since $\dim R/(Rx+\mathfrak{a}) = 1$, $H^f_{Rx+\mathfrak{a}}(M,N)$ is $(Rx+\mathfrak{a})$-cofinite. It follows from the exact sequence ($\S$) and Remark (2.1,iii) that the R-module $H^0_{Rx}(H^f_{\mathfrak{a}}(M,N))$ is $(Rx+\mathfrak{a})$-cofinite. Therefore, the result follows from $H^0_{\mathfrak{b}}(H^0_{Rx}(H^f_{\mathfrak{a}}(M,N)) \cong H^0_{\mathfrak{b}}(H^f_{\mathfrak{a}}(M,N))$ and Theorem 2.5. The last part of the theorem is clear by [15, Theorem 1.1] and the definition of $f_{\mathfrak{a}}(M,N)$. □

3. Cofiniteness of $H^j_{\mathfrak{b}}(H^i_{\mathfrak{a}}(M,N))$ for some indices i, j.

Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R and let $\mathfrak{m}$ be a maximal ideal of R such that $\mathfrak{a} + \mathfrak{b}$ is $\mathfrak{m}$-primary. The aim of this section is to study the cofiniteness of the modules $H^j_{\mathfrak{b}}(H^i_{\mathfrak{a}}(M,N))$ for some particular valuse of "*i's*".

Definition 3.1. Let us define the following number:

$$q_{\mathfrak{a}}(M,N) = \sup\big\{i \mid H^i_{\mathfrak{a}}(M,N) \text{ is not Artinian}\big\}.$$

If $H^i_{\mathfrak{a}}(M,N)$is Artinian for all i, we write $q_{\mathfrak{a}}(M,N) = -\infty$.
In addition, $cd_{\mathfrak{a}}(M,N)$ denotes the largest non-negative integer i such that $H^i_{\mathfrak{a}}(M,N)$ is not equal to zero.

Theorem 3.2. *Let $\mathfrak{a}$, $\mathfrak{b}$ be two ideals of R and let $\mathfrak{m}$ be a maximal ideal of R such that $\mathfrak{a}+\mathfrak{b}$ is $\mathfrak{m}$-primary. If SuppM $\subseteq V(\mathfrak{a})$, then $H^i_{\mathfrak{b}}(M)$ is Artinian and $\mathfrak{b}$-cofinite for all i.*

Proof. It is argued by induction on i. It is straightforward to see that the result is true when $i = 0$. Now, inductively assume that $i > 0$ and that the assertion has been proved for $i - 1$. It follows, from [1, Corollary 2.1.7(iii)] that $H^i_{\mathfrak{b}}(M) \cong H^i_{\mathfrak{b}}\big(M/\Gamma_{\mathfrak{b}}(M)\big)$ for all $i \geq 1$. Also, $M/\Gamma_{\mathfrak{b}}(M)$ is a $\mathfrak{b}$-torsion free R-module. Then the ideal $\mathfrak{b}$ contains an element x, which avoids all members of Ass$M$. It is clear that $\text{Supp}(M/xM) \subseteq V(\mathfrak{a})$. In addition, the exact sequence $0 \longrightarrow M \longrightarrow M \longrightarrow M/xM \longrightarrow 0$ induces the exact sequence

$$H^{i-1}_{\mathfrak{b}}(M/xM) \longrightarrow H^i_{\mathfrak{b}}(M) \xrightarrow{x} H^i_{\mathfrak{b}}(M) \longrightarrow H^i_{\mathfrak{b}}(M/xM),$$

that implies that the R-module $\big(0 :_{H^i_{\mathfrak{b}}(M)} x\big)$ is Artinian and $\mathfrak{b}$-cofinite. Therefore, in view of [10, Proposition 4.1], $H^i_{\mathfrak{b}}(M)$ is Artinian and $\mathfrak{b}$-cofinite. □

Theorem 3.3. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R and $\mathfrak{m}$ be a maximal ideal of R such that $\mathfrak{a}+\mathfrak{b}$ is $\mathfrak{m}$-primary. Let $ara(\mathfrak{b})=t$ and $cd_{\mathfrak{a}}(M,N)=c$. Then $H^t_{\mathfrak{b}}\big(H^c_{\mathfrak{a}}(M,N)\big)$ and $H^{t-1}_{\mathfrak{b}}\big(H^c_{\mathfrak{a}}(M,N)\big)$ are Artinian R-modules.*

Proof. Consider the Grothendieck spectral sequence [11, Theorem 11.38]

$$E_2^{p,q}=H^p_{\mathfrak{b}}\big(H^q_{\mathfrak{a}}(M,N)\big)\overset{p}{\Longrightarrow} H^{p+q}_{\mathfrak{m}}(M,N).$$

This spectral sequence induces an exact sequence of R-modules and R-homomorphisms

$$0\longrightarrow \ker d_2^{i,j}\longrightarrow E_2^{i,j}\overset{d_2^{i,j}}{\longrightarrow} E_2^{i+2,j-1} \text{ for all } i\geq 0. \qquad (\sharp)$$

By the hypotheses $E_2^{p,q}=0$ for all $p>t$ or $q>c$. Then the sequence ($\sharp$) yields the isomorphisms below: $\mathrm{Ker} d_2^{t,c}\cong E_2^{t,c}$, $\mathrm{Ker} d_2^{t-1,c}\cong E_2^{t-1,c}$ and $E_2^{t,c}\cong E_r^{t,c}$ and $E_2^{t-1,c}\cong E_r^{t-1,c}$ for all $r\geq 2$, it follows that $E_\infty^{t-1,c}\cong E_2^{t-1,c}\cong H^{t-1}_{\mathfrak{b}}\big(H^c_{\mathfrak{a}}(M,N)\big)$ and $E_\infty^{t,c}\cong E_2^{t,c}\cong H^t_{\mathfrak{b}}\big(H^c_{\mathfrak{a}}(M,N)\big)$. Now, since $E_\infty^{p,q}$ is a subquotient of the Artinian R-module $H^{p+q}_{\mathfrak{m}}(M,N)$ for each $p,q\in\mathbb{N}_0$, the assertion immediately follows. □

Remark 3.4. Let $(R,\mathfrak{m})$ be a local ring and let $x_1,x_2,\cdots,x_n$ be elements of R. For each $i=1,\cdots,n$, we put $N_i=N/(x_1,x_2,\cdots,x_i)N$ and $\Omega=\{\mathfrak{p}\in \mathrm{Ass}N \mid \dim R/\mathfrak{p}>1\}$. Then the element x_1 is a generalized regular element of N in $\mathfrak{a}$ if $x_1\in\mathfrak{a}-\bigcup_{\mathfrak{p}\in\Omega}\mathfrak{p}$. The sequence $x_1,x_2,\cdots,x_n$ is named to be a generalized regular sequence of N in $\mathfrak{a}$ of length n if x_i is a generalized regular element of N_i in $\mathfrak{a}$ for all $i=1,\cdots,n$. The length of a maximal generalized regular N-sequence in $\mathfrak{a}$ is called the generalized depth of N in $\mathfrak{a}$ and is denoted by $\mathrm{gdepth}(\mathfrak{a},N)$. It is clear that $\mathrm{gdepth}(M/\mathfrak{a}M,N)$ is a non-negative integer and it is equal to the length of any maximal generalized regular N-sequence in $\mathfrak{a}+(0:_R M)$.

Lemma 3.5. *(see [14, Theorem 3.2]). Let $(R,\mathfrak{m})$ be local ring. Then*

$$gdepth(M/\mathfrak{a}M,N)=\min\big\{i \mid Supp H^i_{\mathfrak{a}}(M,N) \text{ is an infinite set}\big\}.$$

Lemma 3.6. *(see [12, Theorem 1.2]). Let t be a non-negative integer such that $\dim Supp\big(H^i_{\mathfrak{a}}(M,N)\big)\leq 1$ for all $i<t$. Then $H^i_{\mathfrak{a}}(M,N)$ is $\mathfrak{a}$-cofinite for all $i<t$.*

Theorem 3.7. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R and let $\mathfrak{m}$ be a maximal ideal of R such that $\mathfrak{a}+\mathfrak{b}$ is $\mathfrak{m}$-primary and let $gdepth(M/\mathfrak{a}M,N)=t$. Then $H^j_{\mathfrak{b}}(H^i_{\mathfrak{a}}(M,N))$ is Artinian and $\mathfrak{b}$-cofinite for all $i<t$ and $j\geq 0$. Moreover, $H^j_{\mathfrak{b}}(H^t_{\mathfrak{a}}(M,N))$ is an Artinian and $\mathfrak{b}$-cofinite R-module for all $j=0,1$.*

Proof. By Lemma 3.6 and Lemma 3.5, we have that $H^i_{\mathfrak{a}}(M,N)$ is $\mathfrak{a}$-cofinite for all $i<t$. It is straightforward that $H^j_{\mathfrak{b}}(H^i_{\mathfrak{a}}(M,N))\cong H^j_{\mathfrak{m}}(H^i_{\mathfrak{a}}(M,N))$. Hence, by Theorem 2.2, $H^j_{\mathfrak{b}}(H^i_{\mathfrak{a}}(M,N))$ is Artinian and $\mathfrak{b}$-cofinite for all $j\geq 0$ and $i<t$. Since, by [11, Theorem 11.38], the Grothendieck's spectral sequence $E_2^{p,q}=H^p_{\mathfrak{b}}\big(H^q_{\mathfrak{a}}(M,N)\big)$ converges to $H^{p+q}_{\mathfrak{m}}(M,N)$. It follows from previous paragraph that $E_2^{p,q}$ is Artinian and $\mathfrak{b}$-cofinite for all $q<t$. Note that $H^i_{\mathfrak{m}}(M,N)$ is Artinian for all $i\geq 0$. By using an argument similar to the proof of [4, Theorem 2.2], we obtain that $H^j_{\mathfrak{b}}(H^t_{\mathfrak{a}}(M,N))$ is Artinian for all $j=0,1$. Since the radical of the annihilator of $\mathrm{Hom}(R/\mathfrak{b},H^j_{\mathfrak{b}}(H^t_{\mathfrak{a}}(M,N))$ is equal to $\mathfrak{m}$, the R-module $H^j_{\mathfrak{b}}(H^t_{\mathfrak{a}}(M,N))$ is Artinian and $\mathfrak{b}$-cofinite for all $j=0,1$. □

Lemma 3.8. *Let $\mathfrak{a}$, $\mathfrak{b}$ be two ideals of R and let $\mathfrak{m}$ be a maximal ideal of R such that $\mathfrak{a}+\mathfrak{b}$ is $\mathfrak{m}$-primary. Let $\Gamma_{\mathfrak{a}}(T)=T$ and we assume that T is an Artinian R-module. Then $H^i_{\mathfrak{b}}(M,T)$ is Artinian and $\mathfrak{b}$-cofinite for all i.*

Proof. The hypotheses says that $\mathrm{Hom}(R/\mathfrak{b},\Gamma_{\mathfrak{b}}(T))$ is of finite length. Therefore, by [10, Proposion 4.1], we deduce that $\Gamma_{\mathfrak{b}}(T)$ is $\mathfrak{b}$-cofinite and Artinian. Now, one can complete the proof by using a similar method which we used in the proof of Theorem 2.2. □

Theorem 3.9. *Let us suppose that there exists an integer $t\geq 0$ such that $H^i_{\mathfrak{a}}(M,N)$ is $\mathfrak{a}$-cofinite for all $i\neq t$. Then $H^i_{\mathfrak{a}}(M,N)$ is $\mathfrak{a}$-cofinite for all i and $H^j_{\mathfrak{b}}\big(H^i_{\mathfrak{a}}(M,N)\big)$ is an Artinian and $\mathfrak{b}$-cofinite R-module for all i, j, where $\mathfrak{a}+\mathfrak{b}$ is $\mathfrak{m}$-primary.*

Proof. Let us consider the convergent spectral sequence

$$E_2^{p,q} = \operatorname{Ext}_R^p\big(R/\mathfrak{a}, H_{\mathfrak{a}}^q(M,N)\big) \overset{p}{\Longrightarrow} \operatorname{Ext}_R^{p+q}\big(M/\mathfrak{a}M, \Gamma_{\mathfrak{a}}(N)\big).$$

Since $E_r^{p,q}$ is a subquotient of $E_2^{p,q}$ for all $r \geq 2$, our hypotheses give us that $E_r^{p,q}$ is finitely generated for all $r \geq 2$, $p \geq 0$, and $q \neq t$. For each $r \geq 2$ and $p, q \geq 0$, let $Z_r^{p,q} = \operatorname{Ker}(E_r^{p,q} \longrightarrow E_r^{p+r,q-r+1})$ and $B_r^{p,q} = \operatorname{Im}(E_r^{p-r,q+r-1} \longrightarrow E_r^{p,q})$. Note that $B_r^{p,q}$ is finitely generated for all p, q and $r \geq 2$, since either $E_r^{p-r,q+r-1}$ or $E_r^{p,q}$ is finitely generated. For all $r \geq 2$ and $p \geq 0$ we have the exact sequences

$$0 \longrightarrow B_r^{p,t} \longrightarrow Z_r^{p,t} \longrightarrow E_{r+1}^{p,t} \longrightarrow 0 \qquad \text{and}$$

$$0 \longrightarrow Z_r^{p,t} \longrightarrow E_r^{p,t} \longrightarrow B_r^{p+r,t-r+1} \longrightarrow 0.$$

On the other hand, $E_\infty^{p,t}$ is isomorphic to a subquotient of $\operatorname{Ext}_R^{p+t}\big(M/\mathfrak{a}M, \Gamma_{\mathfrak{a}}(N)\big)$. Thus it is finitely generated for all p. Since $E_\infty^{p,t} = E_r^{p,t}$ for r sufficiently large,it follows that $E_r^{p,t}$ is finitely generated for all p and all large r. Fix p and r and suppose $E_{r+1}^{p,t}$ is finitely generated. From the first exact sequence we obtain that $Z_r^{p,t}$ is finitely generated. From the second exact sequence we get that $E_r^{p,t}$ is finitely generated for all $r \geq 2$. In particular, $E_2^{p,t} = \operatorname{Ext}_R^p\big(R/\mathfrak{a}, H_{\mathfrak{a}}^t(M,N\big)$ is finitely generated for all p and the result follows from Theorem 2.2. ☐

The following corollary immediately follows from Theorem 3.9 and Definition 3.1.

Corollary 3.10. *Let $\mathfrak{a}$, $\mathfrak{b}$ be two ideals of R and let $\mathfrak{m}$ be a maximal ideal of R such that $\mathfrak{a}+\mathfrak{b}$ is $\mathfrak{m}$-primary. Let $f_{\mathfrak{a}}(M,N) = q_{\mathfrak{a}}(M,N) = t$. Then $H_{\mathfrak{b}}^j\big(H_{\mathfrak{a}}^i(M,N)\big)$ is an Artinian and $\mathfrak{b}$-cofinite R-module for all i, j.*

Proof. If $i < f_{\mathfrak{a}}(M,N)$ then, in view of the definition of $f_{\mathfrak{a}}(M,N)$ and Theorem 3.2, $H_{\mathfrak{b}}^j\big(H_{\mathfrak{a}}^i(M,N)\big)$ is an Artinian and $\mathfrak{b}$-cofinite R-module for all j. If $i > q_{\mathfrak{a}}(M,N)$, then $H_{\mathfrak{a}}^i(M,N)$ is Artinian. It follows from Lemma 3.8 that $H_{\mathfrak{b}}^j\big(H_{\mathfrak{a}}^i(M,N)\big)$ is an Artinian and $\mathfrak{b}$-cofinite R-module for all j. Thus we consider the case where $i = t$. To this end, consider the Grothendieck spectral sequence

$$E_2^{p,q} = H_{\mathfrak{b}}^p\big(H_{\mathfrak{a}}^q(M,N)\big) \overset{p}{\Longrightarrow} H_{\mathfrak{m}}^{p+q}\big(M,N\big).$$

By using an argument similar with that one used in the proof of Theorem 3.9 the result follows. ☐

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References

[1] M. P. Brodmann and R. Y. Sharp, *Local cohomology : An Algebraic introduction with geometric applications*, Cambridge University. Press, 1998.

[2] K. Bahmanpour and R. Naghipur, *Cofiniteness of local cohomology modules for ideals of small dimension*, J. Algebra. **321** (2009), 1997-2011.

[3] F. Dehghani-Zadeh, *Cofinite modules and asymptotic behaviour of generalized local cohomology*, Proceedings of the 5-th Asian Mathematical Conference, Malaysia, (2009), 592-602.

[4] F. Dehghani-Zadeh, *Finiteness properties generalized local cohomology containing the irrelevant ideal*, J. Korean. Math. Soc. **49** (2012), 1215-1228.

[5] D. Delfino, T. Marley, *Cofinite modules and local cohomology*, J. Pure Appl. Algebra.**121**, (1997), 47-52.

[6] A. Grothendieck, *Cohomologie local des faiscfaux coherents et théoreémes de lefschetz locaux et globaux (SGA2)*, North Holland, Amsterdam, 1968.

[7] R. Hartshorne, *Affine duality and cofiniteness*, Invent. Math. **9** (1970), 145-164.

[8] J. Herzog, *Komplexe, Auflösungen und Dualität in der Lokalen Algebra*, Habilitationsschrift, Universität Regensburg, 1974.

[9] L. Melkersson, *Properties of cofinite modules and applications to local cohomology*, Math. Proc. Cambridge Philos. Soc. **125** (1999), 417-423.
[10] L. Melkersson, *Modules cofinite with respect to an ideal*, J. Algebra. **285** (2005), 649-668.
[11] J. Rotman, *An introduction to homological algebra*, Academic Press, New York, 1979.
[12] N. Tu Cuong, S. Goto and N. Van Hong, *On the cofiniteness of generalized local cohomology modules*, Kyoto J. Math. **55** (2015), no. 1, 169-185.
[13] C. A. Weibel, *An introduction to homological algebra*, Cambridge: Cambridge University Press, 1994.
[14] N. Zamani, *Finiteness results on generalized local cohomology modules*. Algebra colloq. **16** (2009), no. 1, 65-70.
[15] K. I. Yoshida, *Cofiniteness of local cohomology modules for dimension on ideals*, Nagoya Math. J. **147** (1995), 179-191.

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