AFFINOID SUBDOMAINS AS COMPLETIONS OF AFFINE SUBDOMAINS

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ABSTRACT. By following an idea of Nicolae Popescu, we construct affinoid subdomains as the completion of affine subdomains.

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1. INTRODUCTION

Throughout this paper all rings are commutative with identity. Let A be a ring and let $A[X_1, \ldots, X_n]$ be the polynomial algebra over A. For simplicity, for any $\nu = (i_1, \ldots, i_n) \in \mathbb{N}^n$, we denote $\mathbf{X}^\nu = X_1^{i_1} \cdots X_n^{i_n}$ and $a_\nu = a_{i_1,\ldots,i_n}$. We also denote $\mathbf{X} = (X_1, \ldots, X_n)$ and $N(\nu) = i_1 + i_2 + \ldots + i_n$. Thus we may write $P \in A[\mathbf{X}]$ as

$$P = \sum_\nu a_\nu \mathbf{X}^\nu,\ a_\nu \in A. \tag{1.1}$$

If $g_1, \ldots, g_n \in A$ and $\nu = (i_1, \ldots, i_n) \in \mathbb{N}^n$, we denote $\mathbf{g}^\nu = g_1^{i_1} \ldots g_n^{i_n}$.

Let A, B be two rings. A homomorphism of rings $\phi : A \to B$ is called an *epimorphism of rings* if for any pair of homomorphisms of rings $\psi_1, \psi_2 : B \to C$, in another arbitrary ring C, the condition $\psi_1\phi = \psi_2\phi$ implies $\psi_1 = \psi_2$. The epimorphism of rings ϕ is called a *flat epimorphism of rings* if the A-module B is flat (see [1], Ch. 1).

The following result is known.

Theorem 1.1. *([4], p. 261) Let $\varphi : A \to B$ be a homomorphism of rings. The following assertions are equivalent:*

a) φ is a flat epimorphism of rings.

b) Let $\mathcal{F} = \{I$ ideal of A such that $\varphi(I)B = B\}$. Then:

i) For any $b \in B$, there exists $I \in \mathcal{F}$ such that $\varphi(I)b \subseteq \varphi(A)$;

ii) If $x \in A$, and $\varphi(x) = 0$, there exists $I \in \mathcal{F}$ such that $Ix = 0$.

If K is a field, a finitely generated K-algebra A is called an *affine K-algebra*. By an *affine subdomain* of Sp $A := (\text{Max}A, A)$, where Max A is the set of maximal ideals of A, we understand a subset $\mathcal{U} \subset$Max$A$ and a homomorphism of affine algebras $\varphi : A \to B$ such that:

i) $\varphi^a(\text{Max } B) \subset \mathcal{U}$, where $\varphi^a(M) := \varphi^{-1}(M)$,

ii) If $\psi : A \to C$ is a homomorphism of affine algebras such that $\psi^a(\text{Max } C) \subset \mathcal{U}$, then there exists a unique homomorphism of affine algebras $\bar{\psi} : B \to C$ such that $\bar{\psi}\phi = \psi$.

Let A be a ring. A function $\| \ \| : A \to [0, \infty)$ is called a *non-archimedean semi-norm* on A if the following properties are satisfied:

i) $\|0\| = 0$,

ii) $\|x - y\| \leq \max\{\|x\|, \|y\|\}$, for all $x, y \in A$,

iii) $\|xy\| \leq \|x\|\|y\|$, for all $x, y \in A$,

iv) $\|1\| \leq 1$.

A non-archimedean semi-norm is called a *non-archimedean norm* if

v) $\|x\| = 0$, $x \in A$, implies $x = 0$.

In this case the pair $(A, \| \|)$ is called a *normed ring*.

Let $(A, \| \|_A)$ be a semi-normed ring (that is $\| \|_A$ is a non-archimedean semi-norm on A). If $P \in A[X_1, ..., X_n]$ is given by (1.1), define the *Gauss semi-norm* of P (see [2], p. 36) by

$$\|P\| = \max_{\nu} \|a_\nu\|_A. \tag{1.2}$$

Throughout this paper the semi-norm on $A[X_1, ..., X_n]$ will be the Gauss semi-norm.

If $(A, \| \|)$ is a semi-normed ring and I be an ideal of A. Denote A/I the quotient ring of A with respect to I and $\pi : A \to A/I$ the natural homomorphism. Then $(A/I, \| \|_{\text{res}})$, where

$$\|\pi(a)\|_{\text{res}} := \inf_{a' - a \in I} \|a'\|, \tag{1.3}$$

is a semi-normed ring. The corresponding topology on A/I is called the *quotient topology*.

Let A and B be two semi-normed rings. A ring homomorphism $\phi : A \to B$ is said to be *strict* if the induced isomorphism $\bar{\phi} : A/\text{Ker}\phi \to \phi(A)$ is a homeomorphism (see [2], p. 21). Here the topology on $A/\text{Ker}\phi$ is the quotient topology and on $\phi(A)$ we consider the induced topology from B.

If $| \ |$ is a non-archimedean norm on A such that $|xy| = |x||y|$, for all $x, y \in A$, then $| \ |$ is called a non-archimedean absolute value (valuation) on A and the pair $(A, | \ |)$ is called a *valued ring*.

Let $(K, | \ |)$ be a valued field and let $A = K[X_1, ..., X_n]/I$ be a K-affine algebra. Throughout this paper we consider on A the quotient topology defined by Gauss norm on $K[X_1, ..., X_n]$.

Let $(K, | \ |)$ be a complete valued field. For a positive integer n the following K-subalgebra of the K-algebra of formal power series in n indeterminates over K (see [2], p. 192):

$$T_n = K < X_1, ..., X_n > := \{ \sum_{i_1, ..., i_n} a_{i_1 ... i_n} X_1^{i_1} ... X_n^{i_n} : a_{i_1 ... i_n} \in K, \lim_{i_1 + ... + i_n \to \infty} |a_{i_1 ... i_n}| = 0 \}$$

is called the *Tate algebra in n indeterminates over K*.

Each residue algebra T_n/I of T_n by an ideal I of T_n is a K-Banach algebra with respect to the residue norm defined by (1.3) (see [2], p. 221). This last K-Banach algebra T_n/I is called a *K-affinoid algebra*.

An *affinoid subdomain* of Sp $A := (\text{Max}A, A)$, where A is a K-affinoid algebra is a subset $\mathcal{U} \subset \text{Max}A$ and a homomorphism of affinoid algebras $\varphi : A \to B$ such that:

i) $\varphi^a(\text{Max } B) \subset \mathcal{U}$, where $\varphi^a(M) := \varphi^{-1}(M)$,

ii) If $\psi : A \to C$ is a homomorphism of affine algebras such that $\psi^a(\text{Max } C) \subset \mathcal{U}$, then there exists a unique homomorphism of affinoid algebras $\bar{\psi} : B \to C$ such that $\bar{\psi}\phi = \psi$.

As a corollary of a theorem of Gerritzen and Grauert (see [2], p. 309) it is known that an affinoid subdomain is a finite union of rational subdomains (defined in [2], p. 282). Moreover, a rational subdomain is constructed as the completion of a suitable ring of fractions (see [2], p. 232). As a continuation of the paper [3] my teacher Nicolae Popescu proposed, about ten years ago, to construct affinoid subdomains as completions of affine domains, which generalize the case when B is a ring of fractions of A. This paper, written to the *memory of Nicolae Popescu* (1937-2010), is a first step in this direction.

The readers are expected to be familiar with the basic notations and results of commutative algebra and non-archimedean analysis, which can be found in, e.g. [5] and [2], respectively.

2. Affine subdomains

Let A be a ring and let $I = (g_1, g_2, ..., g_n)$ be a finitely generated ideal of A. For a fixed non-negative integer m, denote

$$B = A[X_1, ..., X_n]/J,\ J = (\sum_{i=1}^{n} g_i X_i - 1, \mathbf{g}^\nu X_j - a_j^{(\nu)}), \tag{2.1}$$

where are considered all $\nu = (i_1, ..., i_n)$, with $N(\nu) = m$, and $a_j^{(\nu)} \in A$, $j = 1, 2, ..., n$. Denote by $\phi_I : A \to B$ the canonical homomorphism.

In order to give a sufficient condition under which ϕ_I is a flat epimorphism of rings we prove the following result:

Lemma 2.1. *Let A be a ring and let m be a non-negative integer. If, in $A[X_1, \ldots, X_n]$,*

$$\sum_{\substack{\nu = (i_1, ..., i_n) \\ N(\nu) \leq m}} a_\nu (\alpha_1 X_1 - \beta_1)^{i_1} ... (\alpha_n X_n - \beta_n)^{i_n} = 0, \tag{2.2}$$

where $a_\nu, \alpha_j, \beta_j \in A$, $j = 1, 2, ..., n$, then for every $\tau = (j_1, ..., j_n)$, with $N(\tau) = m$ it follows that

$$\alpha^\tau a_\nu = 0, \text{ for all } \nu \text{ with } N(\nu) \leq m,\ \alpha = (\alpha_1, ..., \alpha_n). \tag{2.3}$$

Proof. We use mathematical induction on m. Since (2.3) holds for $m = 0$, assume it holds for $m = s$.

We note that, for every $\nu = (i_1, ..., i_n)$, $\delta = (j_1, ..., j_n)$, with $N(\nu) = m$, $N(\delta) \leq m - 1$, there exist $c_{\delta\nu} \in A$ such that

$$(\alpha_1 X_1 - \beta_1)^{i_1} ... (\alpha_n X_n - \beta_n)^{i_n} = \alpha^\nu \mathbf{X}^\nu + \sum_{\substack{\delta = (j_1, ..., j_n) \\ N(\delta) \leq m-1}} c_{\delta\nu} (\alpha_1 X_1 - \beta_1)^{j_1} ... (\alpha_n X_n - \beta_n)^{j_n}. \tag{2.4}$$

Then, for $m = s + 1$, the equation (2.2) can be written as

$$\sum_{\substack{\tau = (j_1, ..., j_n) \\ N(\tau) = s+1}} a_\tau \alpha^\tau \mathbf{X}^\tau + \sum_{\substack{\nu = (i_1, ..., i_n) \\ N(\nu) \leq s}} a'_\nu (\alpha_1 X_1 - \beta_1)^{i_1} ... (\alpha_n X_n - \beta_n)^{i_n} = 0, \tag{2.5}$$

where

$$a'_\nu = a_\nu + \sum_{\substack{\tau = (j_1, ..., j_n) \\ N(\tau) = s+1}} a_\tau c_{\nu,\tau},\ N(\nu) \leq s,\ c_{\nu,\tau} \in A. \tag{2.6}$$

By (2.5) we get

$$\alpha^\tau a_\tau = 0, \text{ for all } \tau = (j_1, ..., j_n) \text{ with } N(\tau) = s + 1. \tag{2.7}$$

Since (2.3) holds for $m = s$, by equations (2.2), (2.5) and (2.7), it follows that for all $\sigma = (r_1, ..., r_n)$, with $N(\sigma) = s$, we obtain

$$\alpha^\sigma a'_\nu = 0, \text{ for all } \nu \text{ with } N(\nu) \leq s. \tag{2.8}$$

Now, by (2.6)-(2.8), it follows that

$$\alpha^\tau a_\nu = 0, \text{ for all } \nu \text{ with } N(\nu) \leq s + 1, \tag{2.9}$$

which implies the lemma. □

Theorem 2.2. *Let* $I = (g_1, ..., g_n)$ *be an ideal of* A *and let* $a_j^{(\nu)} \in A$*, where* $j = 1, 2, ..., n$*,* $N(\nu) = m$ *and* m *is a fixed positive integer. If there exists* $N \in \mathbb{N}$ *such that for all* τ *with* $N(\tau) = m - 1$*,*

$$I^N(g^\tau - \sum_{j=1}^{n} a_j^{(\tau+\varepsilon^{(j)})}) = 0, \ \varepsilon^{(j)} = (\delta_{1,j}, ..., \delta_{n,j}), \tag{2.10}$$

$$I^N(a_j^{(\tau+\varepsilon^{(s)})} g_r - a_j^{(\tau+\varepsilon^{(r)})} g_s) = 0, \ j, r, s = 1, 2, ..., n, \tag{2.11}$$

then $\phi_I : A \to B$*, where* B *is defined in (2.1), is a flat epimorphism of rings.*

Proof. Let $\mathcal{F} = \{I' : I' \text{ an ideal of } A, \ \varphi_I(I')B = B\}$. Then, by (2.1), $I \in \mathcal{F}$ and, for all $j = 1, 2, ..., n$, $\phi_I(I^m)\bar{X}_j \subset \phi_I(A)$, where $\bar{X}_j$ is the canonical image of X_j in B. Hence it follows that condition b) i) from Theorem 1.1 is fulfilled.

Now we verify condition b) ii) from Theorem 1.1.

If $x \in A$ and $\phi_I(x) = 0$, then, for every $j = 1, 2, ..., n$ and ν, with $N(\nu) = m$, there exist $P, Q_j^{(\nu)} \in A[X_1, ..., X_n]$ such that

$$x = P(\sum_{j=1}^{n} g_j X_j - 1) + \sum_{j=1}^{n} \sum_{\substack{\nu = (i_1, ..., i_n) \\ N(\nu) = m}} Q_j^{(\nu)}(\mathbf{g}^\nu X_j - a_j^{(\nu)}). \tag{2.12}$$

If $\sigma = (r_1, ..., r_n)$, with $N(\sigma) = m$, there exists a positive integer t, and τ with $N(\tau) = m - 1$ such that $\sigma = \tau + \varepsilon^{(t)}$. Hence, by (2.10), $I^N g^\sigma = I^N g^{\tau+\varepsilon^{(t)}} = I^N \sum_{j=1}^{n} a^{(\tau+\varepsilon^{(j)})} g_t$ and

$$I^N \left(\mathbf{g}^\sigma x - P \sum_{j=1}^{n} (\mathbf{g}^\sigma g_j X_j - g_t a_j^{(\tau+\varepsilon^{(j)})}) - \mathbf{g}^\sigma \sum_{j=1}^{n} \sum_{\substack{\nu = (i_1, ..., i_n) \\ N(\nu) = m}} (\mathbf{g}^\nu X_j - a_j^{(\nu)}) Q_j^{(\nu)} \right) = 0. \tag{2.13}$$

Since, by (2.11), $I^N(g_t a_j^{(\tau+\varepsilon^{(j)})} - g_j a_j^{(\sigma)}) = 0$ and $I^N(\mathbf{g}^\sigma a_j^{(\nu)} - \mathbf{g}^\nu a_j^{(\sigma)}) = 0$, by denoting

$$S_j = g_j P + \sum_{\substack{\nu = (i_1, ..., i_n) \\ N(\nu) = m}} \mathbf{g}^\nu Q_j^{(\nu)}, \tag{2.14}$$

the equation (2.13) becomes

$$I^N \left(\mathbf{g}^\sigma x - \sum_{j=1}^{n} (\mathbf{g}^\sigma X_j - a_j^{(\sigma)}) S_j \right) = 0, \text{ for all } \sigma \text{ with } N(\sigma) = m. \tag{2.15}$$

Denote

$$d = \max_{1 \le j \le n} (\deg S_j),$$

which, by (2.14), is independent of ν. Then, by (2.4), for all $\theta = (s_1, ..., s_n)$, with $N(\theta) = ndm + 1$, it follows that there exists $\sigma = (r_1, ..., r_n)$, with $N(\sigma) = m$, such that

$$\mathbf{g}^\theta S_j = \sum_{\substack{\delta = (t_1, ..., t_n) \\ N(\delta) \le d}} b_j^{(\delta)} \left(\mathbf{g}^\sigma X_1 - a_1^{(\sigma)} \right)^{t_1} ... \left(\mathbf{g}^\sigma X_n - a_n^{(\sigma)} \right)^{t_n}, \ b_j^{(\delta)} \in A. \tag{2.16}$$

Since I^N is finitely generated, by (2.15), (2.16) and Lemma 2.1 with $a_{\mathbf{0}} = \mathbf{g}^{\theta+\gamma}$, where $N(\gamma) = N$, it follows that $I^M x = 0$, where $M \geq N + m + ndm + 1$. Thus the condition b) ii) from Theorem 1.1 holds and ϕ_I is a flat epimorphism of rings. □

Corollary 2.3. *Under the hypotheses of Theorem 2.2, for all $I_1 \in \mathcal{F}$, there exists a non-negative integer M such that $I^M \subset I_1$.*

Proof. If $I_1 \in \mathcal{F}$, there exist a positive integer t, $x_i \in I_1$, $b_i \in B$, $i = 1, 2, ..., t$, such that

$$\sum_{i=1}^{t} \varphi_I(x_i) b_i = 1. \tag{2.17}$$

By Theorems 1.1 b) i) and 2.2 we can choose a non-negative integer M_1 such that $\varphi_I(I^{M_1}) b_i \subset \varphi_I(A)$. Hence we get, for all σ, with $N(\sigma) = M_1$,

$$\varphi_I(\mathbf{g}^\sigma) b_i = \varphi_I(\alpha_i^{(\sigma)}),\ \alpha_i^{(\sigma)} \in A. \tag{2.18}$$

By (2.17) and (2.18) it follows that

$$\varphi_I(\mathbf{g}^\sigma) = \sum_{i=1}^{t} \varphi_I(x_i \alpha_i^{(\sigma)}),$$

and by Theorem 2.2 and by Theorem 1.1 b) ii) there exists a non-negative integer M_2 such that, for all σ, with $N(\sigma) = M_1$, we get

$$I^{M_2}(\mathbf{g}^\sigma - \sum_{i=1}^{t} x_i \alpha_i^{(\sigma)}) = 0. \tag{2.19}$$

Since $x_i \in I_1$, by (2.19), it follows that for $M = M_1 + M_2$, $I^M \subset I_1$. □

Example 2.4. Let A be a ring and let $I = (g_1, ..., g_n)$ be an ideal of A. We choose, for example, the elements $b_j^{(s)} \in A$, $j, s = 1, 2, ..., n$, such that $\sum\limits_{j=1}^{n} b_j^{(j)} = 1$, and, for $j \neq s$, $b_j^{(s)} = g_s$. If we take $a_j^{(\tau+\varepsilon^{(s)})} = \mathbf{g}^\tau b_j^{(s)}$, it follows that (2.10) and (2.11) hold. Thus φ_I is a flat epimorphism of rings.

Remark 2.5. Let K be a field and let A be a K-affine algebra. If B is defined by (2.1), then, by Theorem 2.2, $\mathcal{U} = \phi_I^a(\text{Max } B)$, is an affine subdomain of Sp $A = (\mathcal{U}, A)$ (see [3]).

Theorem 2.6. *Let K be a field and let $\phi : A \to B$ be a homomorphism of K-affine algebras such that $\mathcal{U} = \phi^a(\text{Max}B)$ and ϕ define an affine subdomain of* Sp A*. Let $\mathcal{F} = \{I'$ ideal in $A; \phi(I')B = B\}$. If there exists $I \in \mathcal{F}$ such that, for all $I' \in \mathcal{F}$, there exists a positive integer t such that $I^t \subset I'$, then there exist the positive integers n, N, m, and for all $\tau \in \mathbb{N}^n$ with $N(\tau) = m - 1$, $i = 1, 2, ..., n$, there exist $a_i^{(\tau+\varepsilon^{(s)})} \in A$, $s = 1, 2, ..., n$, such that we can take $I = (g_1, ..., g_n)$ such that (2.10), (2.11) hold.*

Proof. Since $\mathcal{U}$ and ϕ define an affine subdomain of Sp A, by Theorem 3.2 from [3], ϕ is a flat epimorphism of rings. Because $I \in \mathcal{F}$ it follows that there exists a positive integer n such that

$$\sum_{i=1}^{n} \phi(g_i) b_i = 1,\ g_i \in I,\ b_i \in B. \tag{2.20}$$

Without loss of generality we may assume $I = (g_1, ..., g_n)$. By Theorem 1.1 b) i) and by hypotheses there exists a positive integer m such that, for all ν with $N(\nu) = m$, we get

$$\phi(g^\nu) b_i = \phi(a_i^{(\nu)}),\ a_i^{(\nu)} \in A. \tag{2.21}$$

If $\tau \in \mathbb{N}^n$ with $N(\tau) = m - 1$, by (2.21),

$$\phi(a_i^{(\tau+\varepsilon^{(r)})}) \phi(g_s) = \phi(g^{\tau+\varepsilon^{(r)}}) b_i \phi(g_s) = \phi(g^{\tau+\varepsilon^{(s)}}) b_i \phi(g_r) = \phi(a_i^{(\tau+\varepsilon^{(s)})}) \phi(g_r),\ r, s = 1, ..., n.$$

Then, by Theorem 1.1 b) ii), there exists a positive integer n_1 such that

$$I^{n_1}(a_j^{(\tau+\varepsilon^{(s)})} g_r - a_j^{(\tau+\varepsilon^{(r)})} g_s) = 0,\ j, r, s = 1, 2, ..., n.$$

Similarly, by (2.20) and (2.21), we get

$$\phi(g^\tau) = \sum_{j=1}^{n} \phi(g^{\tau+\varepsilon^{(j)}}) b_j = \sum_{j=1}^{n} \phi(a_j^{(\tau+\varepsilon^{(j)})}).$$

Then, by Theorem 1.1 b) ii), there exists a positive integer n_2 such that

$$I^{n_2}(g^\tau - \sum_{j=1}^{n} a_j^{(\tau+\varepsilon^{(j)})}) = 0.$$

By taking $N = \max\{n_1, n_2\}$ it follows the statement of the theorem. □

3. Affinoid subdomains

Let K be a complete non-archimedean valued field and let A be a K-affine algebra. We need the following result:

Lemma 3.1. *Let $A = K[Z_1, ..., Z_r]/I_1$ be a K-affine algebra, where I_1 is an ideal of $K[Z_1, ..., Z_r]$. Then $\tilde{A}$ (the completion of A with respect to the residue semi-norm defined by Gauss semi-norme) is an affinoid K-algebra.*

Proof. Since the canonical homomorphism of semi-normed K-affine algebra $\pi_A : K[Z_1, ..., Z_r] \to A$ is a strict homomorphism which is onto, by Corollary 6 from [2], p. 23, we get that $\tilde{\pi}_A : K < Z_1, ..., Z_r > \to \tilde{A}$ is onto. Hence it follows the lemma. □

If I is an ideal of A, denote by A_I the algebra B defined in (2.1).

Theorem 3.2. *Let K be a complete non-archimedean valued field, let A be a K-affine algebra and let I be an ideal of A satisfying the conditions (2.10) and (2.11) (see Theorem 2.6). Then the canonical homomorphism $\tilde{\phi}_I : \tilde{A} \to \tilde{A}_I$ defines the affinoid subdomain $\mathcal{U} = \tilde{\phi}_I^a(\mathrm{Max}\ \tilde{A}_I)$ of $\mathrm{Sp}\ \tilde{A}$.*

Proof. By the canonical commutative diagram

$$\begin{array}{ccc} A[X_1, ..., X_n] & \xrightarrow{\ \pi\ } & A_I \\ \Big\downarrow i_{A[X_1,...X_n]} & & \Big\downarrow i_{A_I} \\ \tilde{A} < X_1, ..., X_n > & \xrightarrow{\ \tilde{\pi}\ } & \tilde{A}_I \end{array} ,$$

where π is a strict homomorphism of rings which is onto and, by Proposition 5 from [2], p. 22, it follows that $\tilde{A}_I \cong \tilde{A} < X_1, ..., X_n > /J\tilde{A} < X_1, ..., X_n >$, because $\tilde{J} = J\tilde{A} < X_1, ..., X_n >$ (see [2], Proposition 3, p. 222).

Let $\psi : \tilde{A} \to C$ be a homomorphism of K-affinoid algebras such $\psi^a(\mathrm{Max}\ C) \subset \tilde{\phi}_I^a(\mathrm{Max}\ \tilde{A}_I)$. We prove that $\psi(I)C = C$.

Suppose the contrary. Then there exists $M_C \in \mathrm{Max}\ C$ such that $\psi(I)C \subset M_C$. Hence $I \subset \psi^a(M_C) = \tilde{\phi}_I^a(M)$, where $M \in \mathrm{Max}\ \tilde{A}_I$. Then $\tilde{\phi}_I(I) \subset M$, a contradiction since $\phi_I(I)A_I = A_I$ implies $\tilde{\phi}_I(I)\tilde{A}_I = \tilde{A}_I$. Thus $\psi(I)C = C$ and there exist $d^{(1)}, ..., d^{(n)} \in C$ such that

$$\sum_{i=1}^{n} \psi(g_i) d^{(i)} = 1. \tag{3.1}$$

We identify $\tilde{A}_I$ with $\tilde{A} < X_1, ..., X_n > /J\tilde{A} < X_1, ..., X_n >$, and, by considering $c^{(i)} = \bar{X}_i$, from (2.1) we get

$$\sum_{i=1}^{n} \tilde{\phi}_I(g_i)c^{(i)} = 1 \tag{3.2}$$

and

$$\tilde{\phi}_I(\mathbf{g}^\nu)c^{(i)} = \tilde{\phi}_I(a_i^\nu),\ i = 1, 2, ..., n,\ \text{for all } \nu \text{ with } N(\nu) = m. \tag{3.3}$$

For an arbitrary positive integer r, by (3.1), it follows that

$$\sum_{\sigma; N(\sigma)=r}^{n} \psi(\mathbf{g}^\sigma)d^{(\sigma)} = 1, \tag{3.4}$$

where $d^{(\sigma)}$ are monomials of degree r in $d^{(1)}, ..., d^{(n)}$ whose coefficients are non-negative integers.

By multiplying (3.1) by $\psi(a_j^{(\tau+\varepsilon^{(j)})}\mathbf{g}^\delta)$, where $N(\tau) = m - 1$, $N(\delta) = N$ and by using (2.11) we find

$$\psi(\mathbf{g}^\delta)\psi(a_j^{(\tau+\varepsilon^{(j)})}) = \sum_{i=1}^{n} \psi(a_j^{(\tau+\varepsilon^{(i)})})\psi(g_j)d^{(i)}\psi(\mathbf{g}^\delta).$$

By multiplying by $d^{(\delta)}$, by summing with respect to δ, with $N(\delta) = N$, and by using (3.4) we get

$$\psi(a_j^{(\tau+\varepsilon^{(j)})}) = \sum_{i=1}^{n} \psi(a_j^{(\tau+\varepsilon^{(i)})})d^{(i)}\psi(g_j),\ \text{for all } \tau \text{ with } N(\tau) = m - 1. \tag{3.5}$$

By multiplying (3.5) by $\psi(\mathbf{g}^\delta)$, by summing with respect to j, and by using (2.10) it follows that

$$\psi(\mathbf{g}^\delta)\psi(\mathbf{g}^\tau) = \psi(\mathbf{g}^\delta)\sum_{j=1}^{n}\sum_{i=1}^{n} \psi(a_j^{(\tau+\varepsilon^{(i)})})d^{(i)}\psi(g_j).$$

Then, by multiplying once again by $d^{(\delta)}$ and by summing with respect to δ, we find

$$\psi(\mathbf{g}^\tau) = \sum_{j=1}^{n}\sum_{i=1}^{n} \psi(a_j^{(\tau+\varepsilon^{(i)})})d^{(i)}\psi(g_j). \tag{3.6}$$

By multiplying (3.6) by $d^{(\tau)}$, with $N(\tau) = m - 1$, and, by using (3.4), we get

$$\sum_{j=1}^{n}\left(\sum_{\tau, N(\tau)=m-1}\sum_{i=1}^{n} \psi(a_j^{(\tau+\varepsilon^{(i)})})d^{(i)}d^{(\tau)}\right)\psi(g_j) = 1. \tag{3.7}$$

If we denote, for $j = 1, 2, ..., n$,

$$\tilde{d}^{(j)} = \sum_{\tau, N(\tau)=m-1}\sum_{i=1}^{n} \psi(a_j^{(\tau+\varepsilon^{(i)})})d^{(i)}d^{(\tau)}, \tag{3.8}$$

then, from (3.7), we find

$$\sum_{j=1}^{n} \psi(g_j)\tilde{d}^{(j)} = 1. \tag{3.9}$$

If $N(\nu) = m$, $N(\delta) = N$, by (2.11) and (3.8), it follows that

$$\psi(\mathbf{g}^{\nu+\delta})\tilde{d}^{(j)} = \sum_{\tau, N(\tau)=m-1}\sum_{i=1}^{n} \psi(a_j^{(\tau+\varepsilon^{(i)})})d^{(i)}d^{(\tau)}\psi(\mathbf{g}^\nu)\psi(\mathbf{g}^\delta)$$

$$= \psi(\mathbf{g}^{\delta}) \sum_{\tau, N(\tau)=m-1} \sum_{i=1}^{n} \psi(a_j^{(\nu)}) \psi(\mathbf{g}^{\tau+\varepsilon^{(i)}}) d^{(\tau)} d^{(i)}$$

$$= \psi(\mathbf{g}^{\delta}) \psi(a_j^{(\nu)}) \sum_{\tau, N(\tau)=m-1} \sum_{i=1}^{n} \psi(\mathbf{g}^{\tau}) d^{(\tau)} \psi(\mathbf{g}^{\varepsilon^{(i)}}) d^{(i)} = \psi(\mathbf{g}^{\delta}) \psi(a_j^{(\nu)}).$$

Hence

$$\psi(\mathbf{g}^{\delta}) \psi(\mathbf{g}^{\nu}) \tilde{d}^{(j)} = \psi(\mathbf{g}^{\delta}) \psi(a_j^{(\nu)}).$$

By multiplying by $d^{(\delta)}$ and by summing with respect to δ, we find

$$\psi(\mathbf{g}^{\nu}) \tilde{d}^{(j)} = \psi(a_j^{(\nu)}),\ j = 1, 2..., n. \tag{3.10}$$

Let $M_C \in$Max C. Then C/M_C is a finite extension of K (see [2], Corollary 3, p. 228) and

$$|\psi(\mathbf{g}^{\nu})|_{C/M_C} = |\mathbf{g}^{\nu}|_{\tilde{A}/\psi^a(M_C)} = |\mathbf{g}^{\nu}|_{\tilde{A}/\tilde{\phi}_I^a(M)} = |\tilde{\phi}_I(\mathbf{g}^{\nu})|_{\tilde{A}_I/M}, \tag{3.11}$$

where $M \in$ Max $\tilde{A}_I$, $|\ |_{C/M_C}$ is the unique absolute value on C/M_C which extends the absolute value on K and $\psi^a(M_C) = \tilde{\phi}_I^a(M)$ (see [2]).

Similarly we get

$$|\psi(a_j^{\nu})|_{C/M_C} = |\tilde{\phi}_I(a_j^{\nu})|_{\tilde{A}_I/M}. \tag{3.12}$$

By (3.3), (3.10)-(3.12) it follows that, for all $M_C \in$Max C,

$$|\tilde{d}^{(j)}|_{C/M_C} = |c^{(j)}|_{\tilde{A}_I/M}. \tag{3.13}$$

Hence (see [2], p. 169 and p. 236)

$$\|\tilde{d}^{(j)}\|_{\sup} \leq |c^{(j)}|_{\sup} \leq 1, \tag{3.14}$$

and the elements $\tilde{d}^{(j)}$ are power bounded (see [2], Proposition 1, p. 240). Then, by using Proposition 4 from [2], p. 222, there exists a continuous mapping $\theta_{\tilde{A}} : \tilde{A} < X_1, ..., X_n > \to C$ such that

$$\theta_{\tilde{A}}(X_j) = \tilde{d}^{(j)} \text{ and } \theta_{\tilde{A}}/\tilde{A} = \psi.$$

By (3.9) and (3.10) we get $J\tilde{A} < X_1, ..., X_n > \subset$ Ker$\theta_{\tilde{A}}$. Thus there exists a continuous mapping $\theta : \tilde{A}_I \to C$ such that

$$\theta \tilde{\phi}_I = \psi. \tag{3.15}$$

If $\theta' \tilde{\phi}_I = \theta \tilde{\phi}_I$, because $\tilde{\phi}_I i_A = i_{A_I} \phi_I$, and ϕ_I is an epimorphism of rings, it follows that $\theta' i_{A_I} = \theta i_{A_I}$. Since $i_{A_I}(A_I)$ is dense in $\tilde{A}_I$ we get $\theta' = \theta$. Hence $\tilde{A}_I$ is an affinoid subdomain of Sp $\tilde{A}$. □

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