SOME RESULT ON EXTREME POINTS IN ORDERED TOPOLOGICAL CONES

ALI HASSANZADEH[1] AND ILDAR SADEQI[2]

Abstract. A cone theoretic Krein-Milman theorem states that in any locally convex T_0 topological cone, every convex compact saturated subset is the compact saturated convex hull of its m-extreme points. In this paper, we prove the Milman theorem in T_0 topological cone, which is a kind of converse of the Krein-Milman theorem. Moreover, we prove some other results about m-extreme points in T_0 topological cones.

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1. Introduction

The branch of order theory called domain theory was initiated in the early 1970s with the pioneering work of Dana S. Scott on a model of untyped lambda-calculus [16]. Progress in this domain rapidly required a lot of material on (non-Hausdorff) topologies. After about 40 years of domain theory, one is forced to recognize that topology and domain theory have been beneficial to each other [5, 7].

One of Klaus Keimel's many mathematical interests is the interaction between order theory and functional analysis. In recent years this has led to the beginnings of a domain-theoretic functional analysis, which may be considered to be a topic within positive analysis in the sense of Jimmie Lawson [11]. In the latter, notions of positivity and order play a key role, as do lower semicontinuity and so T_0 spaces. Some basic functional analytic tools were developed by Roth and Tix and later on Plotkin and Keimel for these structures. Roth has written several papers in this area including his papers [13, 14] on Hahn-Banach type theorems for locally convex cones. Tix in her 1999 Ph.D. thesis gave a domain-theoretic version of these theorems in the framework of d-cones (see [17, 18]). Plotkin subsequently gave another separation theorem, which was incorporated, together with other improvements, into a revised version of Tix's thesis [19, 12]. Finally, Keimel [9] improved the Hahn-Banach theorems to semitopological cones.

The theory of locally convex cones, with applications to Korovkin type approximation theory for positive operators and to vector-measure theory, developed in the books by Keimel and Roth [10] and Roth [15], respectively.

The extreme points of a convex set are of interest primarily because of the Krein-Milman theorem and its generalizations. The Krein-Milman theorem asserts that a compact convex subset K of a locally convex Hausdorff space is the closed convex hull of its extreme points [2].

In 2008, Goubault-Larrecq [6], proved a Krein-Milman type theorem for non-Hausdorff cones (in the sense of Keimel [9]). In fact, he proved the following analogue of the Krein-Milman theorem: in any locally convex T_0 topological cone C, every convex compact saturated subset is the compact saturated convex hull of its extreme points.

The classical Milman theorem (see [3, Theorem 3.66]), states that if X is a locally convex space and B is a nonempty subset of X such that $conv(B)$ is compact, then $Ext(conv(B)) \subset B$. We show that a similar result holds when X is a T_0 topological cone. Finally, as an application, it is observed that, the solutions points of a linear inequality system, under some conditions are m-extreme points.

2. Preliminaries

For convenience of the reader, we give a survey of the relevant materials from [1, 2, 7] and [9], without proofs, thus making our exposition self-contained.

For subsets A of a partially ordered set P we use the following notations:

$$\downarrow A = \{x \in P : x \leq a \text{ for some } a \in A\},$$
$$\uparrow A = \{x \in P : x \geq a \text{ for some } a \in A\}.$$

It is called that A is a lower or upper set, if $\downarrow A = A$ or $\uparrow A = A$, respectively. Upper sets will also be called saturated and $\uparrow A$ will be called the saturation of A.

We denote by $\mathbb{R}_+$ the subset of all nonnegative reals. Further, $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$and $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$. Addition, multiplication and the order are extended to $+\infty$ in the usual way. In particular, $+\infty$ becomes the greatest element and we put $0 \cdot (+\infty) = 0$.

According to [9], a *cone* is a set C, together with two operations $+ : C \times C \to$ C and $\cdot : \mathbb{R}_+ \times C \to C$ and a neutral element $0 \in C$, satisfying the following laws for all $v, w, u \in C$ and $\lambda, \mu \in \mathbb{R}_+$:

$$\begin{aligned} &0 + v = v, && 1v = v, \\ &v + (w + u) = (v + w) + u, && (\lambda\mu)v = \lambda(\mu v), \\ &v + w = w + v, && (\lambda + \mu)v = \lambda v + \mu v, \\ & && \lambda(v + w) = \lambda v + \lambda w. \end{aligned}$$

An *ordered cone* C is a cone endowed with a partial order $\leq$ such that the addition and multiplication by fixed scalars $r \in \mathbb{R}_+$ are order preserving, that is, for all $x, y, z \in C$ and all $r \in \mathbb{R}_+$:

$$x \leq y \Rightarrow x + z \leq y + z \quad \text{and} \quad rx \leq ry.$$

Let us recall that a *linear function* from a cone $(C, +, \cdot)$ to a cone $(C', +, \cdot)$ is a function $f : C \to C'$ such that $f(v + w) = f(v) + f(w)$ and $f(\lambda v) = \lambda f(v)$, for all $v, w \in C$ and $\lambda \in \mathbb{R}_+$.

A subset D of a cone C is said to be *convex*, if for all $u, v \in D$ and $\lambda \in [0, 1], \lambda u + (1 - \lambda)v \in D$. The convex closure of a set D is defined to be the smallest convex set containing D.

For example, $\mathbb{R}^n_+$ is a cone, with the coordinate-wise operations. On $\mathbb{R}_+$, the order is just the usual order $\leq$ of the reals. On $\mathbb{R}^n_+$, it is the coordinate-wise order.

Recall that a partially ordered set $(A, \leq)$ is called directed if for every $a, b \in A$ there exits $c \in A$ with $a, b \leq c$. A partially ordered set $(D, \leq)$ is called a *cpo* if every directed subset A of D has a least upper bound in D. The least upper bound of a directed subset A is denoted by $\sqcup^{\uparrow} A$, and it is also called the directed supremum, or sometimes the limit of A.

Any T_0 space X comes with an intrinsic order, the *specialization order* which is defined by $x \leq y$ if the element x is contained in the closure of the singleton $\{y\}$ or equivalently, if every open set containing x also contains y.

In any T_0 space X, the downward closure $\downarrow E$ is closed whenever E is finite [6, Page 2].

Given any ordering $\leq$ on a set X, there are at least two topologies with $\leq$ as specialization ordering, the coarsest possible one (upper topology : a base is given by the complements of sets of the form $\downarrow E$, for E a finite subset of X) and the finest possible one (Alexandroff topology) (see [7, Section 4.2.2] for more details). Additionally, there are some other interesting topologies in between. An important example of a topology that sits in between is the Scott topology.

Let D be a partially ordered set. A subset A is called Scott closed if it is a lower set and is closed under supremum of directed subsets, as far as these suprema exist. Complements of Scott closed sets are called Scott open. The collection of all Scott opens is a topology, called the *Scott topology* on D [7, Proposition 4.2.18]. We write D_σ for the set D with the Scott topology.

On the extended reals $\overline{\mathbb{R}}$ and on its subsets $\mathbb{R}_+$ and $\overline{\mathbb{R}}_+$ we use the *upper topology*, the only open sets for which are the open intervals $\{s : s > r\}$. This upper topology is T_0, but far from being Hausdorff.

According to [8], a *topological cone* is a cone C with a T_0 topology such that the addition and scalar multiplication are separately continuous, that is:

Scalar multiplication $(r, a) \mapsto ra : \mathbb{R}_+ \times C \to C$ is jointly continuous,

Adition $(a, b) \mapsto a + b : C \times C \to C$ is jointly continuous.

For example, $\mathbb{R}^n_+$ is a topological cone, $\overline{\mathbb{R}}_+$ is also a topological cone, and $\overline{\mathbb{R}}^n_+$ as well. Again, $\overline{\mathbb{R}}_+$ is equipped with its Scott topology [6].

A cone C with a topology is called *locally convex*, if each point has a neighborhood basis of open convex neighborhoods.

Let C be a cone. For any two points x, y of C, let $]x, y[$ be the set of points of the form $r \cdot x + (1-r) \cdot y$, with $0 < r < 1$. It is tempting to call this the open line segment between x and y, however be aware that it is generally not open.

(Extreme Point) Let X be linear space. An extreme point of a convex set $A \subset X$, is a point $x \in A$, with the property that if $x = ty + (1 - t)z$ with $y, z \in A$ and $t \in [0, 1]$, then $y = x$ and or $z = x$. Ext(A) will denote the set of extreme points of A.

Let B be a subset of a T_0 topological cone C, with specialization ordering.

(m-Extreme Point) An m-extreme point of B is any element $x \in B$ that is minimal in B, and such that there are no two distinct points x_1 and x_2 of B such that $x \in]x_1, x_2[$. $Ext_m(B)$ will denote the set of m-extreme points of B.

In the sequel, we shall need the notion of a closed subset of B. This is by definition the intersection of a closed subset of C with B.

(Face) Call a face A of B any non-empty closed subset of B such that, for any $x_1, x_2 \in Q$, if $]x_1, x_2[$ intersects A, then $]x_1, x_2[$ is entirely contained in A.

A cone-theoretic version of the Krein-Milman theorem [6]: Let C be a locally convex T_0 topological cone, Q a convex compact saturated subset of C. Then B is the smallest convex compact saturated subset of C containing the extreme points of B.

3. Main results

The purpose of this section is to study the concept of m-extreme points in a T_0 topological cone, say C. First, we prove some basic results in topological cones.

Proposition 3.1. *Let C be a topological cone and $B \subset C$. Then we have the following statements:*

(i) *If B is a compact set, then $\uparrow B$ is compact.*
(ii) *If B is a convex set, then $\uparrow B$ and $\downarrow B$ are convex, too.*
(iii) *If B is a upper set, then $conv(B)$ is upper set.*
(iv) *$conv(\uparrow B) = \uparrow conv(B)$.*
(v) *$conv(B)$ is not necessarily included $\uparrow B$.*
(vi) *$Ext_m(B^c) \cap B \subset Ext_m(B)$, where B^c indicates smallest compact saturated subset of C containing B, but the inverse is not necessarily true.*

Proof. (i) Let $\mathcal{O}$ be an open cover for $\uparrow B$, so it is an open cover for B and has a finite subcover for B. This finite subcover is subcover for $\uparrow B$, too, and the proof is complete.

(ii) Let $x, y \in \downarrow B$. So there exist $a, b \in B$ such that $x \leq a$ and $y \leq b$. Since scalar multiplication is continuous, so for $0 \leq \lambda \leq 1$, we have $\lambda x \leq \lambda a$ and $(1 - \lambda)y \leq (1 - \lambda)b$. Since $+$ is continuous, hence

$$\lambda x + (1 - \lambda)y \leq \lambda a + (1 - \lambda)y \leq \lambda a + (1 - \lambda)b.$$

Therefore, $\lambda x + (1 - \lambda)y \in \downarrow B$.

(iii) Let $x \in conv(B)$ and $y \in C$ such that $x \leq y$. Then there exists $0 \leq \lambda \leq 1$ and $x_1, x_2 \in B$ such that $x = \lambda x_1 + (1-\lambda)x_2$. Suppose $0 < \lambda < 1$. Since $\lambda x_1 \leq \lambda x_1 + (1-\lambda)x_2$ and $(1-\lambda)x_2 \leq \lambda x_1 + (1-\lambda)x_2$, so $x_1 \leq \frac{1}{\lambda}y$ and $x_2 \leq \frac{1}{1-\lambda}y$. Thus $\frac{1}{\lambda}y, \frac{1}{1-\lambda}y \in B$. Therefore, $y \in conv(B)$ and $conv(B)$ is an upper set.

(iv) Clearly $conv(\uparrow B) \subset \uparrow conv(B)$. Since $conv(B) \subset conv(\uparrow B)$ and $conv(\uparrow B)$ is an upper set so $\uparrow conv(B) \subset conv(\uparrow B)$.

(v), (vi) The proof is trivial. □

Let $conv^c(B)$ indicates smallest convex compact saturated subset of C containing B. Note that $conv^c(B)$ is not necessarily included $\uparrow B$. To see this, let $C = \mathbb{R}^2_+$ and $B = \{(2,1),(2,2),(1,2)\}$, then $conv^c(B) \not\subseteq \uparrow B$.

Now we prove the following cone-theoretic version of the Milman theorem, which is a kind of converse of Krein-Milman theorem.

Theorem 3.2. *Let C be a topological cone such that for each $a \in C$, the operator $T(x) = a + x$ is an open mapping and let B be a nonempty compact subset of C. Then $Ext_m(conv^c(B)) \subset \uparrow B$.*

Proof. Let $x, y \in C$ such that for some $z \in C$, $y = x + z$. By the definition of topological cone, we know that the function $S : b \mapsto x + b : C \to C$ is continuous. So $S(\overline{\{z\}}) \subset \overline{S(z)}$ and then $x \in \overline{\{y\}}$ and so $x \leq y$. Therefore, we show that every m-extreme point of $conv^c(B)$ lies in $B + C$. Since B is compact, it can be covered by a finite number of sets $x_i + C$, where $x_i \in B$ for $i = 1, 2, \ldots, n$. The sets $B_i := conv^c(B \cap (x_i + C))$ are convex and compact, hence $conv(\cup_{i=1}^n B_i)$ are also convex and compact. It follows easily that $conv^c(B) = conv(\cup_{i=1}^n B_i)$. If $e \in Ext_m(conv^c(B))$, then $e = \sum_{i=1}^n \lambda_i b_i$, where $b_i \in B_i$, $\lambda_i \geq 0$ for all $i = 1, 2, \ldots, n$ and $\sum_{i=1}^n \lambda_i = 1$. Now we conclude that, $e = b_i$ for some b_i. This proves that $Ext_m(conv^c(B)) \subset B + C$, therefore $Ext_m(conv^c(B)) \subset \uparrow B$. □

Example 3.3. Let $B = \{(1,1),(2,1),(1,2)\}$. Then

$$conv^c(B) = \uparrow [(2,1),(1,2)]$$

and therefore

$$Ext_m(conv^c(B)) = \{(2,1),(1,2)\}.$$

Remark 3.4. Note that the compactness condition is essential in the above theorem. The following example shows that without this condition the result is no longer true. Suppose that $C = \mathbb{R}^2_+$ and

$$B = \{(x,y) : x + y = 1,\ \frac{1}{4} \leq x < \frac{3}{4},\ \frac{1}{4} \leq y \leq \frac{3}{4}\}.$$

Then the set of B is not compact and $Ext_m(conv^c(B)) \not\subseteq \uparrow B$.

In the sequel, we denote the Euclidean topology on R^n_+ with τ. Let B be a convex subset of $\mathbb{R}^n$. If $F \subset B$ is a half-line face of B, F is called *extreme ray*. $Exra_m(B)$ will denote the minimal of the set of extreme ray of B. In the classical analysis, every nonempty τ-closed convex subset B of $\mathbb{R}^n$ which does not contain any line is the convex hull of its extreme points and extreme rays, which is due to V. Klee. In the sequel, we show that the similar is true in T_0 topological cone $\mathbb{R}^n_+$.

Theorem 3.5. *Let B be a nonempty saturated compact convex subset of topological cone $\mathbb{R}^n_+$. Then B is convex hull of $Ext_m(B)$ and $Exra_m(B)$.*

Proof. Since B is a saturated compact set, so it is the intersection of all finitely generated upper sets [5, Lemma III-5.7.], so B is a τ-closed set and since $\mathbb{R}^2_+$ does not contain any line, hence

$$B = conv(Ext(B) \cup Exra(B)).$$

Since B is an upper set, so

$$B = \uparrow conv(Ext(B) \cup Exra(B)).$$

Since B is τ-closed, we obtain $\uparrow B = \uparrow Min(B)$. To see this, it suffices to show that $B \subset \uparrow Min(B)$. Let $x \in B$, consider the τ-closed set $W = \downarrow x \cap B$ with coordinate-wise order. Since W is closed and bounded, so every descending chain in W has a lower bound. Hence by Zorn's lemma, W contains at least one minimal element, say y. Note that, $y \in Min(B)$ and $y \leq x$, so $x \in \uparrow Min(B)$. It follows that

$$B = \uparrow Min_B(conv(Ext(B) \cup Exra(B)).$$

Therefore,

$$B = \uparrow conv(Min_B(Ext(B) \cup Exra(B)),$$

in other words,

$$B = \uparrow conv(Ext_m(B) \cup Exra_m(B))$$

and the proof is complete. □

Let us illustrate the above theorem with the following example.

Example 3.6. Let $B = \uparrow [(2,1),(1,2)] \subset \mathbb{R}^2_+$, which is a convex saturated compact set. We get $Ext_m(B) = \{(2,1),(1,2)\}$. Note that, $conv(Ext_m(B)) = [(2,1),(1,2)]$. Therefore $\uparrow conv(Ext_m(B)) = B$.

The concept of extreme point can be applicable in the solving of the linear inequality. Suppose that $P = \{x \in \mathbb{R}^n : Ax \geq b\}$, A is a $(m \times n)$-matrix, $b \in \mathbb{R}^m$. Let $y \in P$ and $A'x = b'$ be the subsystem of $Ax = b$ such that y is a unique solution to this subsystem and $rank(A') = rank(A'\ b') = n$. Then y is an extreme point of P [4, Proposition 3.3.1].

Remark 3.7. In the later discussion, let A be a non-negative $(m \times n)$-matrix and $b \in \mathbb{R}^m_+$. Then y is an m-extreme point of P. Indeed, if $w \in P$ and $w \leq y$, then $b' \leq A'w \leq A'y = b'$ (with coordinate-wise order), and $A'w = b'$. Since y is an unique solution to subsystem $A'x = b'$ so, $w = y$, hence $A'w = b'$.

For example, consider in $\mathbb{R}^3$ the set P of the solution to the system $Ax \geq b$:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &\geq 1 \\ x_1 + x_2 + 4x_3 &\geq 1 \\ 2x_1 + x_3 &\geq 2 \\ x_1 + x_2 + x_3 &\geq 1. \end{aligned}$$

Consider the following subsystem $A'x \geq b$:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &\geq 1 \\ x_1 + x_2 + 4x_3 &\geq 1 \\ 2x_1 + x_3 &\geq 2. \end{aligned}$$

Note that, $rank(A') = rank(A'\ b') = 3$ and $y = (0,1,0)$ is the only solution to the system $A'x = b$, and y is an m-extreme point of P.

References

[1] S. Abramsky and A. Jung, *Domain Theory*, In S. Abramsky, D.M. Gabbay and T.S.E. Maibaum editors, handbook of logic in computer science, vol. 3, Clarendon Press, 1994.

[2] C.D. Aliprantis and K.C. Border, *Infinite Dimensional Analysis: A Hitchhikers Guide*, Springer-Verlag, 3rd edition, Heidelberg and New York, 2006.

[3] M. Fabian, P. Habala, P. Hájek, V. Montesinos and V. Zizler, *Banach Space Theory: The Basis for Linear and Nonlinear Analysis*, CMS Books in Mathematics, Canadian Mathematical Society, Springer, 2011.

[4] M. Florenzano and C.L. Van, *Finite Dimensional Convexity and Optimization*, vol. 13: Studies in economic theory, Springer-Verlag, Berlin, 2001.

[5] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove and D.S. Scott, *Continuous Lattices and Domains*, vol. 93, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 2003.

[6] J. Goubault-Larrecq, *A cone theoretic Krein-Milman theorem*, Rapport derecherche LSV-08-18, ENS Cachan, France (2008).

[7] J. Goubault-Larrecq, *Non-Hausdorff Topology and Domain Theory: Selected Topics in Point-Set Topology*, Cambridge University Press, 2013.

[8] K. Keimel, *Topological cones: Foundations for a domain theoretical semantics combining probability and nondeterminism*, Electronic Notes in Theoretical Computer Science **155** (2006), 423-443.

[9] K. Keimel, *Topological cones: Functional analysis in a T_0-setting*, Semigroup Forum **77** (2008), 109-142.

[10] K. Keimel and W. Roth, *Ordered Cones and Approximation*, vol. 1517, Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1992.

[11] J.D. Lawson, *Domains, integration and positive analysis*, Math. Structures Comput. Sci. **14** (2004), no. 6, 815-832.

[12] G. Plotkin, *A domain-theoretic Banach-Alaoglu theorem*, Math. Structures Comput. Sci. **16** (2006), no. 2, 299-311.

[13] W. Roth, *Hahn-Banach type theorems for locally convex cones*, J. Aust. Math. Soc. **68** (2000), no. 1, 104-125.

[14] W. Roth, *Separation properties for locally convex cones*, J. Convex Anal. **9** (2002), no. 1, 301-307.

[15] W. Roth, *Operator-Valued Measures and Integrals for Cone-Valued Functions*, vol. 1964, Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2009.

[16] D.S. Scott, *Continuous lattices. in toposes, algebraic geometry and logic*, (conf., dalhousie univ., halifax, n. s., 1971), Lecture Notes in Math. **274** (1972), 97-136.

[17] R. Tix, *Continuous d-Cones: Convexity and Powerdomain Constructions*, PhD thesis, Technische Universität Darmstadt. Shaker, Aachen, 1999.

[18] R. Tix, *Some results on Hahn-Banach type theorems for continuous d-cones*, Theor. Comput. Sci. **264** (2001), 205-218.

[19] R. Tix, K. Keimel and G. Plotkin, *Semantic domains combining probability and nondeterminism*, Electron. Notes Theor. Comput. Sci. **129** (2005), 1-104.

[1]Department of Mathematics, Sahand University of Technology, Tabriz, Iran
E-mail address: `a_hassanzadeh@sut.ac.ir, ali.hassanzadeh@guest.unimi.it`

[2]Department of Mathematics, Sahand University of Technology, Tabriz, Iran
E-mail address: `esadeqi@sut.ac.ir`