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The only regular inclines are distributive lattices

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Abstract

An incline is an additively idempotent semiring in which the product of two elements is always less than or equal to either factor. This paper proves that the only regular inclines are distributive lattices, which also implies that there is no noncommutative regular incline.

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1. Introduction

Inclines are additively idempotent semirings in which products are less than or equal to either factor. The study of inclines is generally acknowledged to have started by Z.Q. Cao in a series of his papers in the first half of 1980's (see [4]). Nowadays, one may clearly notice a growing interest in developing the algebraic theory of inclines and their numerous significant applications in diverse branches of mathematics and computer science.

Recently, Meenakshi et al. [5] proved that an incline is regular if and only if it is multiplicatively idempotent and that every commutative regular incline is a distributive lattice. Furthermore, Meenakshi et al. [6]-[10] studied generalized inverses, subtractive ideals, homomorphism theorems and quotient inclines in the setting of regular inclines. But one can easily find out that any example of noncommutative regular inclines has never been given in [5]-[10].

The purpose of this paper is to check if there exist any noncommutative regular inclines. This paper proves that every regular incline is commutative, hence the only regular inclines are distributive lattices.

2. Preliminaries

We recall some known definitions and facts.

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A semiring is a nonempty set R together with two operations of addition $+$ and multiplication $\cdot$ such that $(R, +)$ is a commutative semigroup, $(R, \cdot)$ is a semigroup and multiplication distributes over addition from either side. A semiring R is said to be commutative if $ab = ba$ for all $a, b \in R$ (see [2]).

For a semiring R, an element $a \in R$ is said to be additively (resp. multiplicatively) idempotent if $a + a = a$ (resp. $a^2 = a$). R is said to be additively (resp. multiplicatively) idempotent if every element in R is additively (resp. multiplicatively) idempotent.

For a semiring R, an element $a \in R$ is said to be regular if $axa = a$ for some $x \in R$. R is said to be regular if every element in R is regular.

An incline is an additively idempotent semiring R satisfying $x + xy = x + yx = x$ for all $x, y \in R$. On an incline R, a partial order relation $\leqslant$ is defined by $x \leqslant y \Leftrightarrow x + y = y$ for $x, y \in R$. Then $xy \leqslant x$ and $yx \leqslant x$ hold for all $x, y \in R$. For any $x, y, z \in R$, $y \leqslant z$ implies $xy \leqslant xz$ and $yx \leqslant zx$ (see [1]). For an incline R, $I(R)$ denotes the set of all multiplicatively idempotent elements in R, i.e., $I(R) = \{a \in R \mid a^2 = a\}$.

Lemma 2.1 [5]. Let R be an incline and x an element in R.

(1) x is regular if and only if it is multiplicatively idempotent.

(2) R is regular if and only if it is multiplicatively idempotent.

3. Main results

We shall use Lemma 2.1 almost everywhere in this section.

Lemma 3.1. Let R be an incline and x a regular element in R. If $y \in R$, then the following hold.

(1) xy is regular if and only if $yxy = xy$.
(2) yx is regular if and only if $yxy = yx$.
(3) Both xy and yx are regular if and only if $xy = yxy = yx$.

Proof. Note that $y = y + xyx$. In fact, $xyx = (xy)x \leqslant xy \leqslant y$.
(1) If xy is regular, then $yxy = (y + xyx)xy = yxy + xyx^2y = yxy + xyxy = yxy + (xy)^2 = yxy + xy = xy$. Conversely, if $yxy = xy$, then $(xy)^2 = x(yxy) = x(xy) = x^2y = xy$, thus xy is regular.
(2) If yx is regular, then $yxy = yx(y + xyx) = yxy + yx^2yx = yxy + yxyx = yxy + (yx)^2 = yxy + yx = yx$. Conversely, if $yxy = yx$, then $(yx)^2 = (yxy)x = (yx)x = yx^2 = yx$, thus yx is regular.
(3) follows from parts (1) and (2). □

Lemma 3.2. Let R be an incline and x, y two regular elements in R. Then both xy and yx are regular if and only if $xy = yx$.

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Proof. (*Necessity*) follows from Lemma 3.1(3).
(*Sufficiency*) If $xy = yx$, then $(xy)^2 = x(yx)y = x(xy)y = x^2y^2 = xy$, thus xy and yx are regular. □

Corollary 3.3. Every regular incline is commutative.

Lemma 3.4. If R is a commutative incline and $I(R) \neq \emptyset$, then $I(R)$ is a distributive lattice with respect to the incline operations.

Proof. For any $x, y \in I(R)$, we have $(x+y)^2 = x^2 + yx + xy + y^2 = x + yx + xy + y = x + y$ and $(xy)^2 = xyxy = x^2y^2 = xy$. Hence $x + y \in I(R)$, $xy \in I(R)$ and $x + y = \sup\{x, y\}$ in $I(R)$. If $x \geqslant z$ and $y \geqslant z$ for some $z \in I(R)$, then $xy \geqslant z^2 = z$, which shows that $xy = \inf\{x, y\}$ in $I(R)$. Thus $I(R)$ is a distributive lattice. □

Lemma 3.4 is a slight modification of Lemma 2.3 in [3].

Theorem 3.5. An incline R is regular if and only if it is a distributive lattice with respect to the incline operations.

Proof. If R is regular, then it is commutative by Corollary 3.3. By Lemma 3.4, $I(R)$ is a distributive lattice with respect to the incline operations, provided $I(R) \neq \emptyset$. By Lemma 2.1(2), R is multiplicatively idempotent and so $R = I(R)$. Thus R is a distributive lattice. The converse is obvious. □

The following example shows that in a noncommutative incline, the product of two regular elements is not necessarily regular and its regularity also depends on the arrangement of factors.

Example 3.6. Let $R = \{a, b, c, d, f\}$ be a set of five distinct elements and define two operations $+$ and $\cdot$ on R as follows.

$+$	a	b	c	d	f
a	a	a	a	a	a
b	a	b	a	b	b
c	a	a	c	c	c
d	a	b	c	d	d
f	a	b	c	d	f

$\cdot$	a	b	c	d	f
a	a	b	c	d	f
b	b	b	d	d	f
c	c	f	c	f	f
d	d	f	d	f	f
f	f	f	f	f	f

By computation, one can easily see that R is a noncommutative incline, in which both b and c are regular since they are multiplicatively idempotent. But the product $bc = d$ is not regular because $d^2 = f \neq d$, while $cb = f$ is regular because $f^2 = f$.

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