DISTANCE ROMAN DOMINATION IN RANDOM GRAPHS

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Abstract. For a positive integer k, a subset $D \subseteq V(G)$ is called a *distance-k dominating set* of G if every vertex in $V(G) - D$ is within distance k from some vertex of D. The minimum cardinality among all distance-k dominating sets of G is called the *distance-k domination number* of G. For any positive integer r, a function $f : V(G) \to \{0, 1, 2\}$ is a *Roman r-dominating function* if every vertex u for which $f(u) = 0$ is adjacent to at least r vertices v for which $f(v) = 2$. The weight of a Roman r-dominating function is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The *Roman r-domination number* of a graph G is the minimum weight of a Roman r-dominating function on G. We study distance-k domination number and Roman r-domination number in Random graphs by considering a combined variant namely distance-k Roman r-domination number.

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1. Introduction

Let $G = (V, E)$ be a finite, undirected and simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The number of vertices $|V|$ is called the order of G and is denoted by $n = n(G)$. We denote the *open neighborhood* of a vertex v of G by $N_G(v)$, or just $N(v)$, and its *closed neighborhood* by $N_G[v]$ or $N[v]$. For a vertex set $S \subseteq V(G)$, we denote $N(S) = \cup_{v \in S} N(v)$ and $N[S] = \cup_{v \in S} N[v]$. The *degree* of a vertex x, $\deg(x)$ (or $\deg_G(x)$ to refer G) in a graph G denotes the number of neighbors of x in G. We refer $\delta(G)$ as the *minimum degree* of the vertices of G. A set of vertices S in G is a *dominating set*, if $N[S] = V(G)$. The *domination number*, $\gamma(G)$, of G is the minimum cardinality of a dominating set of G. For references and also terminology on domination in graphs see for example [10, 12].

For a graph G, let $f : V(G) \to \{0, 1, 2\}$ be a function, and let (V_0, V_1, V_2) be the ordered partition of $V(G)$ induced by f, where $V_i = \{v \in V(G) : f(v) = i\}$ and for $i = 0, 1, 2$. There is a $1-1$ correspondence between the functions $f : V(G) \to \{0, 1, 2\}$ and the ordered partition (V_0, V_1, V_2) of $V(G)$. So we will write $f = (V_0, V_1, V_2)$. A function $f : V(G) \to \{0, 1, 2\}$ is a *Roman dominating function* (RDF) if every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of an RDF f is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The *Roman domination number* of a graph G, denoted by $\gamma_R(G)$, is the minimum weight of an RDF on G. Roman domination numbers have been studied, for example, in [4, 17, 18].

For a positive integer k, a subset $D \subseteq V(G)$ is called a *distance-k dominating set* of G if every vertex in $V(G) - D$ is within distance k from some vertex of D. The minimum cardinality among all distance-k dominating sets of G is called the *distance-k domination number* of G. In this paper we denote the distance-k domination number of G by $\gamma^k(G)$. The concept of distance-k domination in graphs was introduced by Henning et al. [11] and further studied for example in [8, 15, 16, 19, 20]. Fink and Jacobson

[6, 7] introduced the concept of r-domination for a positive integer r. A subset $D \subseteq V(G)$ is called an *r-dominating set* of G if every vertex in $V(G) - D$ is adjacent to at least r vertices of D. The minimum cardinality among all r-dominating set of G is called the *r-domination number* of G and is denoted by $\gamma_r(G)$. This concept was further studied, for example in [3, 5, 21, 22].

Kammerling and Volkmann [14] extended the concept of Roman domination to *Roman r-domination*, for any positive integer r. A function $f : V(G) \to \{0,1,2\}$ is a *Roman r-dominating function* if every vertex u for which $f(u) = 0$ is adjacent to at least r vertex v for which $f(v) = 2$. The weight of a Roman r-dominating function is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The *Roman r-domination number* of a graph G, denoted by $\gamma_{rR}(G)$, is the minimum weight of a Roman r-dominating function on G.

Several authors studied domination parameters in Random graphs, see for example [1, 2, 13, 23]. Our aim in this paper is to study the concepts of Roman r-domination and distance-k domination in Random graphs. For this purpose we define a new invariant namely distance-k Roman r-domination which is a generalization of Roman r-domination and distance-k domination. A function $f : V(G) \to \{0,1,2\}$ is a *distance-k Roman r-dominating function* if every vertex u for which $f(u) = 0$ is within distance k of at least r vertex v for which $f(v) = 2$. The weight of a distance-k Roman r-dominating function is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The *distance-$k$ Roman r-domination number* of a graph G, denoted by $\gamma_R^{(k,r)}(G)$, is the minimum weight of a distance-k Roman r-dominating function on G. It is obvious that $\gamma_R^{(1,r)}(G) = \gamma_{rR}(G)$. Also if a graph G has a distance-k Roman 1-dominating function $f = (V_0, V_1, V_2)$ with $V_1 = \emptyset$, then $\gamma_R^{(k,1)}(G) \geq 2\gamma^k(G)$, and thus $\gamma^k(G) = \frac{1}{2}\gamma_R^{(k,1)}(G)$, since clearly $\gamma_R^{(k,1)}(G) \leq 2\gamma^k(G)$. Throughout this paper we assume that $r < \frac{n}{2}$.

2. Main results

Let n be a positive integer and $0 < p < 1$. The *random graph* $G(n,p)$ is a probability space over the set of graphs on the vertex set $[n] = \{1, ..., n\}$ determined by $Pr[\{i,j\} \in E(G)] = p$ with these events mutually independent. We say that an event holds *asymptotically almost surely* (a.a.s.) if the probability that it holds tends to 1 as n tends to infinity. Note that by definition the weigh of any distance-k Roman r-dominating set must be at least $2r$. It is well known that for constant $p < 1$, the diameter of $G(n,p)$ is two a.a.s. Thus if p is constant and $k \geq 2$ then a.a.s. $\gamma_R^{(2,r)}(G(n,p)) = 2r$. The case p constant and $k = 1$ will be addressed as an open problem. We next assume that p is not constant.

Theorem 2.1 (Bollobas, [2]). *Let c be a positive constant, $d = d(n) \geq 2$ a natural number, and define $p = p(n,c,d)$, $0 < p < 1$, by $p^d n^{d-1} = \log(n^2/c)$. Suppose that $pn/(\log n)^3 \to \infty$. Then in $G(n,p)$, we have*

$$(1) \quad \lim_{n\to\infty} \quad Pr(diam\ G = d) = e^{-c/2},$$
$$(2) \quad \lim_{n\to\infty} \quad Pr(diam\ G = d+1) = 1 - e^{-c/2}.$$

From Theorem 2.1, the following can be obtained readily.

Theorem 2.2. *For any positive integers $k \geq 3$ and r, in a random graph $G(n,p)$ with $p = \sqrt[k]{\frac{log(n^2/c)}{n^{k-1}}}$, a.a.s $\gamma_R^{(k,r)}(G(n,p)) = 2r$.*

Next we consider the case $k = 2$.

Theorem 2.3 (Hopcraft and Kannan, [13]). *Let $p = c\sqrt{\frac{\ln n}{n}}$. For $c > \sqrt{2}$, $G(n,p)$ almost surely has diameter less than or equal to two.*

From Theorem 2.3 for $p \geq \sqrt{2}\sqrt{\frac{\ln n}{n}}$ we obtain that $\gamma_R^{(k,r)}(G(n,p)) = 2r$ a.a.s. We will weaken the minimum value of p from $\sqrt{2}\sqrt{\frac{\ln n}{n}}$ to $p \geq c\sqrt{\frac{\ln n}{n}}$, for a fixed constant $c > 1$.

Theorem 2.4. *Let $c>1$ be a fixed constant. For any positive integer r, in a random graph $G(n,p)$ with $p \geq c\sqrt{\frac{\ln n}{n}}$, a.a.s. $\gamma_R^{(2,r)}(G(n,p)) = 2r$.*

Proof. Let $D \subseteq V(G(n,p))$ be a subset with $|D| = r$. Let the vertices in D be labeled as $v_1, v_2, \ldots, v_r$. The probability that a vertex $u \in V(G(n,p)) \setminus D$ is not within distance-2 from a vertex $v_i \in D$ is given by $Pr(u \notin N_2(v_i)) \leq (1-p^2)^{n-2}$. Let X be a random variable that denotes the number of vertices $u \in V(G(n,p)) \setminus D$, where the number vertices of D within distance 2 from u is less than r. We show that $Pr(X>0) \to 0$ as $n \to \infty$.

A fixed vertex u is defined *bad*, if there is less than r vertices in D within distance two from u. By the linearity property of the expectation we have

$$E(X) = (n-r)Pr(\text{fixed } u \text{ is bad}). \tag{2.1}$$

Let X_u be a random variable that denotes the number of vertices in D that are not within distance two from u. Then $E(X_u) \leq r(1-p^2)^{n-2} \leq re^{-p^2(n-2)}$. By the Markov's inequality we have $Pr(X_u > 0) \leq E(X_u) \leq re^{-p^2(n-2)}$. Thus,

$$Pr(\text{fixed } u \text{ is bad}) = Pr(X_u > 0) \leq re^{-p^2(n-2)}. \tag{2.2}$$

By (2.1) and (2.2), we have $E(X) \leq (n-r)re^{-p^2(n-2)}$. By the Markov's inequality we obtain,

$$Pr(X>0) \leq E(X) \leq (n-r)re^{-p^2(n-2)} < nre^{-p^2(n-2)}. \tag{2.3}$$

Since $n \to \infty$, in (2.3), we have $e^{p^2(n-2)} > rn$ for sufficiently large n. This implies that $p^2(n-2) > \ln rn$ and so $p > \sqrt{\frac{\ln rn}{n-2}}$. We conclude that $p > \sqrt{\frac{\ln n}{n}}$. Let $p > c\sqrt{\frac{\ln n}{n}}$, where $c>1$ is a constant. We determine the value of $e^{p^2(n-2)}$.

$$e^{p^2(n-2)} \geq (e^{\ln n})^{c^2(\frac{n-2}{n})} \geq n^{c^2(1-\frac{2}{n})}. \tag{2.4}$$

From (2.3) and (2.4) we have

$$nre^{-p^2(n-2)} \leq \frac{nr}{n^{c^2(1-\frac{2}{n})}} = \frac{r}{n^{c^2(1-\frac{2}{n})-1}}. \tag{2.5}$$

Since $c^2 > 1$ as $n \to \infty$, $c^2(1-\frac{2}{n}) > 1$, and hence, $c^2(1-\frac{2}{n}) - 1 > 0$. Thus, as $n \to \infty$,

$$\frac{r}{n^{c^2(1-\frac{2}{n})-1}} \to 0.$$

Therefore, from (2.3) and (2.5) we have $Pr(X>0) \to 0$ as $n \to \infty$. □

Thus the remaining case is $k=1$. We propose the following problem.

Problem 2.5. *For $k=1$ and $p \in (0,1)$ (p is not necessarily constant) determine $\gamma_R^{(1,r)}(G(n,p))$ a.a.e.*

3. Concluding remarks

We end the paper with stating some probabilistic bounds for the distance-k Roman r-domination number in graphs using similar results on Roman domination and r-domination numbers. It is obvious that $\gamma_R^{(k,r)}(G) = 2r$ if $diam(G) \leq k$. Thus we assume that $diam(G) > k$. For a vertex v let $N_k(v)$ be the set of all vertices u such that $u \neq v$ and is within distance-k from v, and let $\delta_k = \delta_k(G) = \min\{N_k(v) : v \in V(G)\}$. We also define the k-graph G^k as the graph with vertex set $V(G^k) = V(G)$, and $E(G^k) = \{xy : d_G(x,y) \leq k\}$. Note that $G^1 = G$. Hansberg and Volkmann, [9] proved that if G is a graph on n vertices with $\delta(G) \geq r$, where r is a positive integer, and $\frac{\delta(G)+1+2\ln 2}{\ln(\delta(G)+1)} \geq 2r$, then $\gamma_{rR}(G) \leq \left(\frac{2r\ln(\delta(G)+1)-\ln 4+2}{\delta(G)+1}\right)n$. It is obvious that $\gamma_R^{(k,r)}(G) = \gamma_{rR}(G^k)$. Thus from the above upper bound and with an identical proof as the proof of Theorem 11 of [23], we obtain the following.

Theorem 3.1. *If* $\frac{\delta_k+1+2\ln 2}{\ln(\delta_k+1)} \geq 2r$ *and* $\delta_k \geq r$*, then*

$$\gamma_R^{(k,r)}(G) \leq \Big(\frac{2r\ln(\delta_k+1) - \ln 4 + 2}{\delta_k+1}\Big)n.$$

This bound is asymptotically best possible.

References

[1] N. Alon and J. Spencer, *The Probabilistic Method*, John Wiley, New York, (1992).
[2] B. Bollobas, *Random Graphs*, Cambridge University Press (2001).
[3] B. Chen and S. Zhou, Upper bounds for f-domination number of graphs, *Discrete Math.* 185 (1998), 239-243.
[4] E. J. Cockayne, P. M. Dreyer Jr., S. M. Hedetniemi, and S. T Hedetniemi, Roman domination in graphs, *Discrete Math.* 278 (2004), 11-22.
[5] O. Favaron, A. Hansberg and L. Volkmann, On k-domination and minimum degree in graphs, *J. Graph Theory* 57 (2008), 33-40.
[6] J. F. Fink and M. S. Jacobson, *n-domination in graphs Graph Theory with Applications to Algorithms and Computer Science*, John Wiley and Sons, New York, 1985, pp. 283-300.
[7] J. F. Fink and M. S. Jacobson, *On n-domination, n-dependence and forbidden subgraphs*, Graph Theory with Applications to Algorithms and Computer Science, John Wiley and Sons, New York, 1985, pp. 301-311.
[8] A. Hansberg, D. Meierling and L. Volkmann, Distance Domination and Distance Irredundance in Graphs, *Elec. J. Combin.* (2007), R35.
[9] A. Hansberg and L. Volkmann, Upper bounds on the k-domination number and the k-Roman domination number, *Discrete Appl. Math.* 157 (2009), 1634–1639.
[10] T. W. Haynes, S. T. Hedetniemi, and P. J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, (1998).
[11] M. A. Henning, O. R. Oellermann, and H. C. Swart, Bounds on distance domination parameters, *J. Combin. Inform. System Sci.* 16 (1991), 11–18.
[12] M.A. Henning, and A. Yeo, *Total domination in graphs*, Springer Monographs in Mathematics, Springer, New York, (2013).
[13] J. Hopcroft, and R. Kannan, Mathematics for modern computing, Preprint (2013).
[14] K. Kammerling and L. Volkmann, Roman k-domination in graphs, *J. Korean Math. Soc.* 46 (2009), No. 6, pp. 1309-1318.
[15] J. Raczek, M. Lemanska, and J. Cyman, Lower bound on the distance k-domination number of a tree, *Math. Slovaca* 56 (2006), 235–243.
[16] J. Raczek, Distance paired domination numbers of graphs, *Discrete Math.* 308 (2008), 2473–2483.
[17] C. S. ReVelle, and K. E. Rosing, Defendens imperium romanum: a classical problem in military strategy, *Amer. Math. Monthly* 107 (2000), 585-594.
[18] I. Stewart, Defend the Roman Empire!, *Sci. Amer.* 281 (1999), 136-139.
[19] N. Sridharan, V. S.A. Subramanian and M.D. Elias, Bounds on the distance two-domination number of a graph, *Graphs and Combin.* 18 (2002), 667-675.
[20] F. Tian and J. M. Xu, A note on distance domination number of graphs, *Australas. J. Combin.* 43 (2009), 181-190.
[21] L. Volkmann, A bound on the k-domination number of a graph, *Czech. Math. J.* 60 (2010), 77–83.
[22] S. Zhou, Inequalities involving independence domination, f-domination, connected and total f-domination numbers, *Czech. Math. J.* 50 (2000), 321-330.
[23] V. Zverovich, A. Poghosyan, On Roman, Global and Restrained Domination in Graphs, *Graphs and Combin.* 27 (2011), 755–768.

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