

ON THE UPPER BOUND OF THE ENERGY OF A CONNECTED GRAPH

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ABSTRACT. New upper bounds for the energy of a connected graph are presented in this note. The upper bounds involve the independence number of the graph.

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1. INTRODUCTION

All the graphs considered in this note are undirected graphs without loops or multiple edges. Notation and terminology not defined here follow those in [1]. Let G be a graph of order n with e edges. We use $\delta = \delta(G)$ and $\Delta = \Delta(G)$ to denote the minimum and maximum degrees of G , respectively. The independence number, denoted $\alpha = \alpha(G)$, is defined as the size of the largest independent set in G . The 2 - degree, denoted $t(v)$, of a vertex v in G is defined as the sum of degrees of vertices adjacent to v . We use $T = T(G)$ to denote the maximum 2 - degree of G . Obviously, $T(G) \leq (\Delta(G))^2$. A bipartite graph G is called semiregular if all the vertices in the same vertex part of a bipartition of the vertex set of G have the same degree. The eigenvalues $\mu_1(G) \geq \mu_2(G) \geq \dots \geq \mu_n(G)$ of the adjacency matrix $A(G)$ of G are called the eigenvalues of G . The spread, denoted $Spr(G)$, of G is defined as $\mu_1(G) - \mu_n(G)$. The energy, denoted $Eng(G)$, of G is defined as $\sum_{i=1}^n |\mu_i(G)|$ (see [7]).

Several authors have obtained the upper bounds for the energy of a graph (see [5], [8], [9], [12], [13]). In this note, we will present new upper bounds for the energy of a connected graph. The results are as follows.

Theorem 1. *Let G be a connected graph with $n \geq 2$ vertices and e edges. Then*

$$Eng(G) \leq 2\sqrt{e} + 2\sqrt{(n - \alpha - 1) \left(e + \sqrt{T \left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor} - \frac{2\delta^2\alpha}{n - \alpha} \right)}$$

with equality if and only if G is $K_{1,1}$ or $K_{1,2}$.

Obviously, Theorem 1 has the following corollary.

Corollary 1. *Let G be a connected graph with $n \geq 2$ vertices and e edges. Then*

$$Eng(G) \leq 2\sqrt{e} + 2\sqrt{(n - \alpha - 1) \left(e + \Delta \sqrt{\left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor} - \frac{2\delta^2\alpha}{n - \alpha} \right)}$$

with equality if and only if G is $K_{1,1}$.

2. LEMMAS

In order to prove Theorem 1, we need the following lemmas. Lemma 1 below is Theorem 3.14 on Pages 88 and 89 in [4].

Lemma 1. *Let G be a graph. If the number of eigenvalues of G which are greater than, less than, and equal to zero are p , q , and r , respectively, then*

$$\alpha \leq r + \min\{p, q\},$$

where α is the independence number of G .

Lemma 2 below is Theorem 1.5 on Page 26 in [6].

Lemma 2. *For a graph G with n vertices and e edges,*

$$\text{Spr}(G) \leq \mu_1 + \sqrt{2e - \mu_1^2} \leq 2\sqrt{e}.$$

Equality holds throughout if and only if equality holds in the first inequality; equivalently, if and only if $e = 0$ or G is $K_{a,b}$ for some a, b with $e = ab$ and $a + b \leq n$.

Lemma 3 below is obvious.

Lemma 3. *If $x \geq 0$ and $y \geq 0$, then $\sqrt{x} + \sqrt{y} \leq \sqrt{2(x+y)}$ with equality if and only if $x = y$.*

Lemma 4 below is Corollary 3.4 on Page 2731 in [10].

Lemma 4. *Let G be a graph. Then $\text{Spr}(G) \geq 2\delta\sqrt{\frac{\alpha(G)}{n-\alpha(G)}}$. If equality holds, then G is a semiregular bipartite graph.*

Lemma 5 is Theorem 1 on Page 5 in [2].

Lemma 5. *Let G be a connected graph. Then $\mu_1 \leq \sqrt{T(G)}$ with equality if and only if G is either a regular graph or a semiregular bipartite graph.*

Lemma 6 follows from Proposition 2 on Page 174 in [3].

Lemma 6. *Let G be a graph. Then $\mu_n \geq -\sqrt{\lceil \frac{n}{2} \rceil \lfloor \frac{n}{2} \rfloor}$ with equality if and only if G is $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$.*

3. PROOFS

Next, we will present proofs for Theorem 1.

Proof of Theorem 1. Let $\mu_1 \geq \mu_2 \geq \dots \geq \mu_p$ be the p positive eigenvalues of G and let $\rho_q \geq \rho_{q-1} \geq \dots \geq \rho_1$ be the q negative eigenvalues of G . Then G has $n - p - q$ eigenvalues which are equal to zero. From Lemma 1, we have

$$\alpha \leq (n - p - q) + \min\{p, q\}.$$

Thus $\alpha \leq (n - p - q) + q$ and $\alpha \leq (n - p - q) + p$. Namely, $p \leq n - \alpha$ and $q \leq n - \alpha$. Since $\sum_{i=1}^p \mu_i + \sum_{i=1}^q \rho_i = 0$, we have that

$$\text{Eng}(G) = 2 \sum_{i=1}^p \mu_i = 2 \sum_{i=1}^q |\rho_i|.$$

From Cauchy - Schwarz inequality, we have that

$$\frac{Eng(G)}{2} = \sum_{i=1}^p \mu_i \leq \mu_1 + \sqrt{(p-1) \sum_{i=2}^p \mu_i^2} = \mu_1 + \sqrt{(p-1) \left(\sum_{i=1}^p \mu_i^2 - \mu_1^2 \right)}.$$

Similarly, we have that

$$\frac{Eng(G)}{2} = \sum_{i=1}^q |\rho_i| \leq |\rho_1| + \sqrt{(q-1) \sum_{i=2}^q \rho_i^2} = |\rho_1| + \sqrt{(q-1) \left(\sum_{i=1}^q \rho_i^2 - \rho_1^2 \right)}.$$

Hence we get that

$$\begin{aligned} Eng(G) &= \frac{Eng(G)}{2} + \frac{Eng(G)}{2} \\ &\leq \mu_1 + \sqrt{(p-1) \left(\sum_{i=1}^p \mu_i^2 - \mu_1^2 \right)} + |\rho_1| + \sqrt{(q-1) \left(\sum_{i=1}^q \rho_i^2 - \rho_1^2 \right)}. \end{aligned}$$

Then by Lemmas 2 and 3 it follows that

$$\begin{aligned} Eng(G) &\leq 2\sqrt{e} + \sqrt{n-\alpha-1} \left(\sqrt{\left(\sum_{i=1}^p \mu_i^2 - \mu_1^2 \right)} + \sqrt{\left(\sum_{i=1}^q \rho_i^2 - \rho_1^2 \right)} \right) \\ &\leq 2\sqrt{e} + \sqrt{n-\alpha-1} \sqrt{2 \left(\sum_{i=1}^p \mu_i^2 - \mu_1^2 + \sum_{i=1}^q \rho_i^2 - \rho_1^2 \right)}. \end{aligned}$$

Since $\sum_{i=1}^p \mu_i^2 + \sum_{i=1}^q \rho_i^2 = \text{the trace of } A^2 = \text{the sum of diagonal entries of } A^2 = \text{the sum of degrees of vertices in } G = 2e$, we get that

$$\begin{aligned} Eng(G) &\leq 2\sqrt{e} + \sqrt{2(n-\alpha-1)(2e - \mu_1^2 - \rho_1^2)} \\ &= 2\sqrt{e} + \sqrt{2(n-\alpha-1)(2e - (\mu_1 - \rho_1)^2 - 2\mu_1\rho_1)}. \end{aligned}$$

Then by Lemmas 4, 5, and 6 we get that

$$\begin{aligned} Eng(G) &\leq 2\sqrt{e} + \sqrt{2(n-\alpha-1) \left(2e + 2\sqrt{T\left[\frac{n}{2}\right] \lfloor \frac{n}{2} \rfloor} - \frac{4\delta^2\alpha}{n-\alpha} \right)} \\ &= 2\sqrt{e} + 2\sqrt{(n-\alpha-1) \left(e + \sqrt{T\left[\frac{n}{2}\right] \lfloor \frac{n}{2} \rfloor} - \frac{2\delta^2\alpha}{n-\alpha} \right)}. \end{aligned}$$

If G is $K_{1,1}$ or $K_{1,2}$, it is trivial to verify that

$$Eng(G) = 2\sqrt{e} + 2\sqrt{(n-\alpha-1) \left(e + \sqrt{T\left[\frac{n}{2}\right] \lfloor \frac{n}{2} \rfloor} - \frac{2\delta^2\alpha}{n-\alpha} \right)}.$$

If

$$Eng(G) = 2\sqrt{e} + 2\sqrt{(n-\alpha-1) \left(e + \sqrt{T\left[\frac{n}{2}\right] \lfloor \frac{n}{2} \rfloor} - \frac{2\delta^2\alpha}{n-\alpha} \right)},$$

then, from the proofs above, we have that $p = q = n - \alpha$ and $G = K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$. Since G is connected, its adjacency matrix is irreducible. From Perron - Frobenius theorem, we have that $p = 1$ (see [11]). Thus $\alpha = n - 1$. Hence G must be $K_{1,1}$ or $K_{1,2}$. \square

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