

FINITENESS PROPERTIES OF FORMAL LOCAL HOMOLOGY MODULES

M.H. BIJAN-ZADEH, S. GHADERI

ABSTRACT. Let (R, \mathfrak{m}) be a commutative Noetherian ring, \mathfrak{a} an ideal of R and M an Artinian R -module. In this paper, we investigate the structure of the formal local homology. We prove several results concerning finiteness properties of formal local homology module.

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1. INTRODUCTION

Throughout this paper, we assume that (R, \mathfrak{m}) is a commutative Noetherian local ring which have non-zero identity. Let \mathfrak{a} be an ideal of R and M an R -module. It is well known that the \mathfrak{a} -adic completion functor $\Lambda_{\mathfrak{a}}$ is defined by $\Lambda_{\mathfrak{a}}(M) = \varprojlim_t M/\mathfrak{a}^t M$ (see [8], [9]). In [4], N. T. Coung and T. T. Nam defined the local homology module $H_i^{\mathfrak{a}}(M)$ with respect to \mathfrak{a} by

$$H_i^{\mathfrak{a}}(M) = \varprojlim_n \mathrm{Tor}_i^R(R/\mathfrak{a}^n, M).$$

If we use $L_i^{\mathfrak{a}}(M)$ to denote the i -th left derived module of $\Lambda_{\mathfrak{a}}(M)$, then $H_i^{\mathfrak{a}}(M) = L_i^{\mathfrak{a}}(M)$ for the Artinian R -modules (see [4]). For an Artinian R -module M , in [2], the i -th formal local homology module of M with respect to \mathfrak{a} is defined by $\varinjlim_n H_i^{\mathfrak{a}}((0 :_M \mathfrak{a}^n))$ and it is investigated its structure. While the formal local cohomology modules are studied in great detail not so much is known about the formal local homology modules (see [1], [5], [7]).

In this paper, for an integer i and an Artinian R -module M , let $\mathfrak{F}_i^{\mathfrak{a}}(M)$ be the formal local homology module of M with respect to \mathfrak{a} . Let t be a non-negative integer. It is shown that the local homology module $H_i^{\mathfrak{a}}(M)$ is Artinian for all $i < t$ if and only if there is some non-negative integer s such that $\mathfrak{a}^s H_i^{\mathfrak{a}}(M) = 0$ for all $i < t$, provided that M be Artinian R -module (see [4, 4.7]).

We generalize the above result for formal local homology modules. In fact the purpose of this paper is to answer the following question for the formal local homology modules: When are the formal local homology modules finitely generated?

The main aim of this paper is to prove that under some conditions, $\mathfrak{F}_i^{\mathfrak{a}}(M)$ is finitely generated for all $i > t$ if and only if $\mathfrak{a} \subseteq \text{Rad}(\text{Ann}(\mathfrak{F}_i^{\mathfrak{a}}(M)))$ for all $i > t$.

We define the formal finiteness dimension and obtain some finiteness properties of the formal local homology modules. We also show that under some conditions, if $\mathfrak{F}_i^{\mathfrak{a}}(M)$ is finitely generated for all $i > t$ then $\text{Hom}(R/\mathfrak{a}, \mathfrak{F}_i^{\mathfrak{a}}(M))$ is finitely generated.

Throughout this paper, for an R -module M , $E(R/\mathfrak{m})$ denotes the injective envelope of R/\mathfrak{m} and $D(M)$ denotes the Matlis duality functor $\text{Hom}(M, E(R/\mathfrak{m}))$.

2. MAIN RESULTS

Let $\underline{x} = x_1, \dots, x_r$ be a system of elements of R , and $b = \text{Rad}(\underline{x})$. Let $C_{\underline{x}}$ denote the Čech complex of R with respect to \underline{x} , (see [3], [6]). For an R -module M and an ideal \mathfrak{a} , the direct system of R -modules $\{(0 :_M \mathfrak{a}^n)\}_{n \in \mathbb{N}}$ induces a direct system of R -complexes $\{\text{Hom}(C_{\underline{x}}, (0 :_M \mathfrak{a}^n))\}_{n \in \mathbb{N}}$. In [2], the formal local homology module is defined by the following isomorphisms

$$H_i(\varinjlim_n \text{Hom}(C_{\underline{x}}, (0 :_M \mathfrak{a}^n))) \simeq \varinjlim_n H_i^b((0 :_M \mathfrak{a}^n)),$$

for all $i \in \mathbb{Z}$, provided that M be an Artinian R -module.

Notation. Let \mathfrak{a} be an ideal of a local ring (R, \mathfrak{m}) , M be an Artinian R -module and $\mathfrak{m} = \text{Rad}(\underline{x})$. We call $\mathfrak{F}_i^{\mathfrak{a}}(M) := \varinjlim_n H_i^{\mathfrak{m}}((0 :_M \mathfrak{a}^n))$ the i -th \mathfrak{a} -formal local homology.

Definition 2.1. Let M be an Artinian R -module. For an ideal \mathfrak{a} of R , we define the formal finiteness dimension, $ff^{\mathfrak{a}}(M)$ by

$$ff^{\mathfrak{a}}(M) := \inf\{i \mid \mathfrak{F}_i^{\mathfrak{a}}(M) \text{ is not finitely generated}\}.$$

Proposition 2.2. Let $\mathfrak{a}, \mathfrak{b}$ be two ideals of R and M be an Artinian R -module. If $\text{Rad}(\mathfrak{a}) = \text{Rad}(\mathfrak{b})$, then $ff^{\mathfrak{a}}(M) = ff^{\mathfrak{b}}(M)$.

Proof. By assumption $\text{Rad}(\mathfrak{a}) = \text{Rad}(\mathfrak{b})$, it is easy to prove that if $n \in \mathbb{N}$, then there is a positive integer m such that $(0 :_M \mathfrak{a}^n) = (0 :_M \mathfrak{b}^m)$. So the result follows by the definition of the formal local homology. \square

Theorem 2.3. Let \mathfrak{a} be an ideal of R and M be an Artinian R -module. Then $\mathfrak{F}_i^{\mathfrak{a}}(M) \simeq \widehat{\mathfrak{F}}_i^{\mathfrak{a}}(M)$ for all $i \in \mathbb{Z}$, where $\widehat{}$ is the completion functor with respect to \mathfrak{m} .

Proof. It is a consequence of theorem [2, 2.6] for the particular case $\mathfrak{b} = \mathfrak{m}$. \square

Proposition 2.4. Let \mathfrak{a} be an ideal of R and M be an Artinian R -module. Then $ff^{\mathfrak{a}}(M) = ff^{\widehat{\mathfrak{a}}}(M)$, where $\widehat{}$ is the completion functor with respect to \mathfrak{m} .

Proof. It immediately follows from Theorem 2.3. \square

Theorem 2.5. Let \mathfrak{a} be an ideal of R . Let M be an Artinian R -module and $x \in \mathfrak{m}$. Then there is the long exact sequence

$$\dots \rightarrow \widehat{\mathfrak{F}}_i^{(\mathfrak{a}, x)}(M) \rightarrow \mathfrak{F}_i^{\mathfrak{a}}(M) \rightarrow R_x \otimes \mathfrak{F}_i^{\mathfrak{a}}(M) \rightarrow \dots$$

for all $i \in \mathbb{Z}$.

Proof. It is a consequence of theorem [2, 2.13] for the particular case $\mathfrak{b} = \mathfrak{m}$. \square

Corollary 2.6. *Let x be an element of \mathfrak{m} and M be an Artinian R -module. Then there is a short exact sequence*

$$\cdots \rightarrow \varinjlim_n H_i^{\mathfrak{m}}((0 :_M x^n)) \rightarrow H_i^{\mathfrak{m}}(M) \rightarrow R_x \otimes H_i^{\mathfrak{m}}(M) \rightarrow \cdots$$

for all $i \in \mathbb{Z}$.

Proof. Whenever $\mathfrak{a} = 0$, it is a consequence of Theorem 2.5. \square

Theorem 2.7. *Let \mathfrak{a} be an ideal of R and M be an Artinian R -module. Choose $x \in R$ such that $x \notin \mathfrak{a}$. Then $ff^{\mathfrak{a}}(M) \leq ff^{(\mathfrak{a}, x)}(M) + 1$.*

Proof. By Theorem 2.5, there is the following long exact sequence

$$\cdots \rightarrow R_x \otimes \mathfrak{F}_{i+1}^{\mathfrak{a}}(M) \rightarrow \mathfrak{F}_i^{(\mathfrak{a}, x)}(M) \rightarrow \mathfrak{F}_i^{\mathfrak{a}}(M) \rightarrow \cdots$$

For all $i < ff^{\mathfrak{a}}(M) - 1$, $\mathfrak{F}_{i+1}^{\mathfrak{a}}(M)$ and $\mathfrak{F}_i^{\mathfrak{a}}(M)$ are finitely generated. Therefore $\mathfrak{F}_i^{(\mathfrak{a}, x)}(M)$ is finitely generated and this completes the proof. \square

Proposition 2.8. *Let M be an Artinian R -module. Then the R -module $H_i^{\mathfrak{m}}(M)$ is finitely generated for all $i \geq 0$.*

Proof. Since $H_i^{\mathfrak{m}}(M) \simeq H_i^{\hat{\mathfrak{m}}}(M)$ as \hat{R} -module, so without loss of generality we may assume that R is a complete local ring. By Matlis duality, $D(M)$ is finitely generated. Therefore $H_{\mathfrak{m}}^i(D(M))$ is Artinian (see [3, 7.1.3]). But by [4, 3.3(ii)] follows that

$$H_i^{\mathfrak{m}}(M) \simeq H_i^{\mathfrak{m}}(D(D(M))) \simeq D(H_{\mathfrak{m}}^i(D(M))).$$

Thus $H_i^{\mathfrak{m}}(M)$ is a finitely generated R -module. \square

We recall the concept of Krull dimension of an Artinian R -module M , denoted by $\text{Kdim}M$. Let M be an Artinian R -module. When $M = 0$ we put $\text{Kdim}M = -1$. Then by induction, for any ordinal α , we put $\text{Kdim}M = \alpha$ when (i) $\text{Kdim}M < \alpha$ is false, and (ii) for every ascending chain, $M_0 \subseteq M_1 \subseteq \cdots$ of submodules of M , there exists a positive integer m_0 such that $\text{Kdim}(M_{m+1}/M_m) < \alpha$ for all $m \geq m_0$ (see [10]).

For an ideal \mathfrak{a} of (R, \mathfrak{m}) , we can define the formal homological dimension of M with respect to \mathfrak{a} , by

$$f\text{-cd}(\mathfrak{a}, M) = \sup\{i \in \mathbb{Z} : \mathfrak{F}_i^{\mathfrak{a}}(M) \neq 0\}.$$

Theorem 2.9. *Let \mathfrak{a} be an ideal of (R, \mathfrak{m}) . Then $\text{Kdim}(0 :_M \mathfrak{a}) = \sup\{i \in \mathbb{Z} : \mathfrak{F}_i^{\mathfrak{a}}(M) \neq 0\}$, for any non-zero Artinian R -module M .*

Proof. Since $\text{Kdim}(0 :_M \mathfrak{a}^n) = \text{Kdim}(0 :_M \mathfrak{a})$ for all $n \in \mathbb{N}$, by [4, 4.8], $\mathfrak{F}_i^{\mathfrak{a}}(M) = 0$ for all $i > \text{Kdim}(0 :_M \mathfrak{a})$. Therefore

$$\text{Kdim}(0 :_M \mathfrak{a}) \geq \sup\{i \in \mathbb{Z} : \mathfrak{F}_i^{\mathfrak{a}}(M) \neq 0\}.$$

In order to prove the equality, put $\text{Kdim}(0 :_M \mathfrak{a}) = s$. First let we consider the short exact sequence

$$0 \rightarrow (0 :_M \mathfrak{a}^n) \rightarrow (0 :_M \mathfrak{a}^{n+1}) \rightarrow \frac{(0 :_M \mathfrak{a}^{n+1})}{(0 :_M \mathfrak{a}^n)} \rightarrow 0.$$

Because of $\text{Kdim}((0 :_M \mathfrak{a}^{n+1})/(0 :_M \mathfrak{a}^n)) < s$, $H_{s+1}^{\mathfrak{m}}((0 :_M \mathfrak{a}^{n+1})/(0 :_M \mathfrak{a}^n)) = 0$ that induces a monomorphism $0 \rightarrow H_s^{\mathfrak{m}}((0 :_M \mathfrak{a}^n)) \rightarrow H_s^{\mathfrak{m}}((0 :_M \mathfrak{a}^{n+1}))$ of non-zero R -modules for all $n \in \mathbb{N}$ (see [4, 4.2, 4.10]). So $\mathfrak{F}_s^{\mathfrak{a}}(M) := \varinjlim H_s^{\mathfrak{m}}((0 :_M \mathfrak{a}^n)) \neq 0$ and the claim is proved. \square

Theorem 2.10. *Let \mathfrak{a} denote an ideal of R . Let M be an Artinian R -module such that $\text{Kdim } M = d$. Then $\mathfrak{F}_d^{\mathfrak{a}}(M)$ is finitely generated.*

Proof. Let $\mathfrak{a} := (x_1, x_2, \dots, x_n)$. We argue by induction on n . Let $n = 1$. By Corollary 2.6, we have the exact sequence

$$\cdots \rightarrow R_{\mathfrak{a}} \otimes H_{d+1}^{\mathfrak{m}}(M) \rightarrow \mathfrak{F}_d^{\mathfrak{a}}(M) \xrightarrow{\alpha} H_d^{\mathfrak{m}}(M) \rightarrow \cdots$$

By [4, 4.8], $H_{d+1}^{\mathfrak{m}}(M) = 0$. So $\mathfrak{F}_d^{\mathfrak{a}}(M) \simeq \text{Im } \alpha \subseteq H_d^{\mathfrak{m}}(M)$ and the claim follows from Proposition 2.8. Now suppose, inductively, that the result has been proved for $n - 1$ and let we put $\mathfrak{b} = (x_1, x_2, \dots, x_{n-1})$. By Theorem 2.5, there is a long exact sequence

$$\cdots \rightarrow R_{x_n} \otimes \mathfrak{F}_{d+1}^{\mathfrak{b}}(M) \rightarrow \mathfrak{F}_d^{\mathfrak{a}}(M) \xrightarrow{\beta} \mathfrak{F}_d^{\mathfrak{b}}(M) \rightarrow \cdots$$

So, from Theorem 2.9, $\mathfrak{F}_{d+1}^{\mathfrak{b}}(M) = 0$ and $\mathfrak{F}_d^{\mathfrak{a}}(M) \simeq \text{Im } \beta \subseteq \mathfrak{F}_d^{\mathfrak{b}}(M)$. Therefore the induction hypothesis yields that $\mathfrak{F}_d^{\mathfrak{a}}(M)$ is finitely generated. \square

Lemma 2.11. *Let \mathfrak{a} denote an ideal of R , M an Artinian R -module and $N \subset M$ be a summand of M such that $\text{Supp}(M/N) \subseteq V(\mathfrak{a})$. Then there is a natural isomorphism $\mathfrak{F}_i^{\mathfrak{a}}(M/N) \simeq H_i^{\mathfrak{m}}(N)$ for all i and there exists a long exact sequence*

$$\cdots \rightarrow H_{i+1}^{\mathfrak{m}}(M/N) \rightarrow \mathfrak{F}_i^{\mathfrak{a}}(N) \rightarrow \mathfrak{F}_i^{\mathfrak{a}}(M) \rightarrow H_i^{\mathfrak{m}}(M/N) \rightarrow \cdots$$

Proof. Since $\text{Supp}(M/N) \subseteq V(\mathfrak{a})$, it follows that M/N is annihilated by some power of \mathfrak{a} . So

$$\mathfrak{F}_{i+1}^{\mathfrak{a}}(M/N) \simeq \varinjlim_n H_i^{\mathfrak{m}}((0 :_{M/N} \mathfrak{a}^n)) \simeq \varinjlim_n H_i^{\mathfrak{m}}(M/N) \simeq H_i^{\mathfrak{m}}(M/N),$$

for all i . By [2, 2.9], the short exact sequence $0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0$ induces a long exact sequence

$$\cdots \rightarrow \mathfrak{F}_{i+1}^{\mathfrak{a}}(M/N) \rightarrow \mathfrak{F}_i^{\mathfrak{a}}(N) \rightarrow \mathfrak{F}_i^{\mathfrak{a}}(M) \rightarrow \mathfrak{F}_i^{\mathfrak{a}}(M/N) \rightarrow \cdots$$

and, with the isomorphisms above, the claim is proved for all i . \square

Theorem 2.12. *Let M be an Artinian R -module and t be a non-negative integer, such that $\mathfrak{a} \subseteq \text{Rad}(\text{Ann}(\mathfrak{F}_i^{\mathfrak{a}}(M)))$ for all $i > t$. If $\Lambda_{\mathfrak{a}}(M)$ is projective then $\mathfrak{F}_i^{\mathfrak{a}}(M)$ is finitely generated for all $i > t$.*

Proof. We use induction on $\text{Kdim } M = d$. Let $d = 0$. $\mathfrak{F}_i^{\mathfrak{a}}(M) = 0$ for all $i > 0$, so in this case the claim holds. Now let $d > 0$ and suppose that the claim holds for all value less than d . Since M is Artinian, there exists a positive integer s such that $\mathfrak{a}^t M = \mathfrak{a}^s M$ for all $t \geq s$. So $\bigcap_{t \geq 0} \mathfrak{a}^t M = \mathfrak{a}^s M$, $\Lambda_{\mathfrak{a}}(M) = M/\mathfrak{a}^s M$ and we have a short exact sequence of Artinian R -modules

$$0 \rightarrow \bigcap \mathfrak{a}^t M \rightarrow M \rightarrow \Lambda_{\mathfrak{a}}(M) \rightarrow 0.$$

By Lemma 2.11 we get a long exact sequence of local homology modules

$$\cdots \rightarrow H_{i+1}^m(\Lambda_{\mathfrak{a}}(M)) \rightarrow \mathfrak{F}_i^{\mathfrak{a}}(\bigcap \mathfrak{a}^t M) \rightarrow \mathfrak{F}_i^{\mathfrak{a}}(M) \rightarrow H_i^m(\Lambda_{\mathfrak{a}}(M)) \rightarrow \cdots$$

So by Proposition 2.8 it is enough to prove that $\mathfrak{F}_i^{\mathfrak{a}}(\bigcap \mathfrak{a}^t M)$ is finitely generated for all $i > t$. We can replace M by $\bigcap \mathfrak{a}^t M$ and we may assume that $\mathfrak{a}M = M$. Since M is Artinian, $xM = M$ for some $x \in \mathfrak{a}$. Thus, by the hypothesis, there exists a positive integer r such that $x^r \mathfrak{F}_i^{\mathfrak{a}}(M) = 0$ for all $i > t$. Then by [2, 2.9] the short exact sequence

$$0 \rightarrow (0 :_M x^r) \rightarrow M \xrightarrow{x^r} M \rightarrow 0$$

induces an exact sequence

$$0 \rightarrow \mathfrak{F}_{i+1}^{\mathfrak{a}}(M) \rightarrow \mathfrak{F}_i^{\mathfrak{a}}((0 :_M x^r)) \rightarrow \mathfrak{F}_i^{\mathfrak{a}}(M) \rightarrow 0,$$

for all $i > t$. So, by the inductive hypothesis, $\mathfrak{F}_i^{\mathfrak{a}}((0 :_M x^r))$ is finitely generated for all $i > t$. We see that $\mathfrak{F}_i^{\mathfrak{a}}(M)$ is finitely generated for all $i > t$. This finishes the inductive step. \square

Theorem 2.13. *Let \mathfrak{a} be an ideal of R and M be an Artinian R -module. Assume that the integer t is such that $\mathfrak{F}_i^{\mathfrak{a}}(M)$ is finitely generated for all $i > t$. If $\Lambda_{\mathfrak{a}}(M)$ is projective then $(0 :_{\mathfrak{F}_i^{\mathfrak{a}}(M)} \mathfrak{a})$ is finitely generated.*

Proof. We use induction on $\text{Kdim} M = n$. For $n = 0$ we have that $\mathfrak{F}_i^{\mathfrak{a}}(M) = 0$ for all $i > 0$ and $\mathfrak{F}_0^{\mathfrak{a}}(M)$ is finitely generated by Theorem 2.10. Now let $n > 0$ and suppose the claim holds for all values less than n . By the same argument as in the proof of Theorem 2.12, there is an element $x \in \mathfrak{a}$, such that $xM = M$. Then the short exact sequence of Artinian modules

$$0 \rightarrow (0 :_M x) \rightarrow M \xrightarrow{x} M \rightarrow 0,$$

implies that

$$\cdots \rightarrow \mathfrak{F}_{i+1}^{\mathfrak{a}}(M) \rightarrow \mathfrak{F}_i^{\mathfrak{a}}((0 :_M x)) \rightarrow \mathfrak{F}_i^{\mathfrak{a}}(M) \rightarrow \cdots$$

It yields that $\mathfrak{F}_i^{\mathfrak{a}}((0 :_M x))$ is finitely generated for all $i > t$. Thus, by the induction hypothesis, $(0 :_{\mathfrak{F}_i^{\mathfrak{a}}((0 :_M x))} \mathfrak{a})$ is finitely generated. Now consider the exact sequence

$$\cdots \rightarrow \mathfrak{F}_{t+1}^{\mathfrak{a}}(M) \xrightarrow{g} \mathfrak{F}_t^{\mathfrak{a}}((0 :_M x)) \xrightarrow{f} \mathfrak{F}_t^{\mathfrak{a}}(M) \xrightarrow{x} \mathfrak{F}_t^{\mathfrak{a}}(M) \rightarrow \cdots$$

which breaks into two short exact sequences

$$0 \rightarrow \text{Im} g \rightarrow \mathfrak{F}_t^{\mathfrak{a}}((0 :_M x)) \rightarrow \text{Im} f \rightarrow 0,$$

$$0 \rightarrow \text{Im} f \rightarrow \mathfrak{F}_t^{\mathfrak{a}}(M) \xrightarrow{x} \mathfrak{F}_t^{\mathfrak{a}}(M).$$

The first of these sequences induces a long exact sequence

$$\cdots \rightarrow (0 :_{\mathfrak{F}_t^{\mathfrak{a}}((0 :_M x))} \mathfrak{a}) \rightarrow (0 :_{\text{Im} f} \mathfrak{a}) \rightarrow \text{Ext}^1(R/\mathfrak{a}, \text{Im} g) \cdots$$

where $(0 :_{\mathfrak{F}_t^{\mathfrak{a}}((0 :_M x))} \mathfrak{a})$ and $\text{Ext}^1(R/\mathfrak{a}, \text{Im} g)$ are finitely generated. The second sequence induces a long exact sequence

$$0 \rightarrow (0 :_{\text{Im} f} \mathfrak{a}) \rightarrow (0 :_{\mathfrak{F}_t^{\mathfrak{a}}(M)} \mathfrak{a}) \xrightarrow{x} (0 :_{\mathfrak{F}_t^{\mathfrak{a}}(M)} \mathfrak{a}) \cdots$$

Since $x \in \mathfrak{a}$, it follows that $(0 :_{\text{Im} f} \mathfrak{a}) \simeq (0 :_{\mathfrak{F}_t^{\mathfrak{a}}(M)} \mathfrak{a})$. This completes the proof. \square

Corollary 2.14. *Let M be an Artinian R -module. Then $(0 :_{\mathfrak{F}_{f-cd(\mathfrak{a},M)}^{\mathfrak{a}}(M)} \mathfrak{a})$ is finitely generated.*

Proof. It immediately follows from Theorem 2.13. □

Theorem 2.15. *Let \mathfrak{a} be an ideal of R and M be an Artinian R -module. Let t be a non-negative integer. Then the following statements are equivalent:*

- (a) $\mathfrak{m} \subseteq \text{Rad}(\text{Ann}(\mathfrak{F}_i^{\mathfrak{a}}(M)))$ for all $i > t$,
- (b) $\mathfrak{F}_i^{\mathfrak{a}}(M)$ is finitely generated for all $i > t$,
- (c) $\mathfrak{F}_i^{\mathfrak{a}}(M) = 0$ for all $i > t$.

Proof. (a) \Rightarrow (b). We use induction on $\text{Kdim}M = d$. Let $d = 0$. $\mathfrak{F}_i^{\mathfrak{a}}(M) = 0$ for all $i > 0$, so in this case the claim holds. Now let $d > 0$ and suppose that the claim hold for all value less than d . Since $(0 :_M \mathfrak{a}^n)$ is Artinian, we can replace it by $\bigcap_{t>0} \mathfrak{m}^t(0 :_M \mathfrak{a}^n)$ (see [4, 4.5]). We can assume that $\mathfrak{m}(0 :_M \mathfrak{a}^n) = (0 :_M \mathfrak{a}^n)$ and since M is Artinian, $x(0 :_M \mathfrak{a}^n) = (0 :_M \mathfrak{a}^n)$ for some $x \in \mathfrak{m}$. From the hypothesis it follows that there exists a positive integer s such that $x^s \mathfrak{F}_i^{\mathfrak{a}}(M) = 0$ for all $i > t$. Thus for all $i > t$ the exact sequence

$$0 \rightarrow (0 :_{(0 :_M x^s)} \mathfrak{a}^n) \rightarrow (0 :_M \mathfrak{a}^n) \xrightarrow{x^s} (0 :_M \mathfrak{a}^n) \rightarrow 0,$$

implies that

$$0 \rightarrow \mathfrak{F}_{i+1}^{\mathfrak{a}}(M) \rightarrow \mathfrak{F}_i^{\mathfrak{a}}((0 :_M x^s)) \rightarrow \mathfrak{F}_i^{\mathfrak{a}}(M) \rightarrow 0.$$

So, by the inductive hypothesis, $\mathfrak{F}_i^{\mathfrak{a}}((0 :_M x^s))$ is finitely generated for all $i > t$, and we see that $\mathfrak{F}_i^{\mathfrak{a}}(M)$ is finitely generated for all $i > t$.

(b) \Rightarrow (c). We use induction on $\text{Kdim}M = d$. Let $d = 0$. $\mathfrak{F}_i^{\mathfrak{a}}(M) = 0$ for all $i > 0$, so in this case the claim holds. Now let $d > 0$ and suppose that the claim hold for all value less than d . As we did in the proof of (a) \Rightarrow (b), there is the following exact sequence

$$\cdots \rightarrow \mathfrak{F}_{i+1}^{\mathfrak{a}}(M) \rightarrow \mathfrak{F}_i^{\mathfrak{a}}((0 :_M x^s)) \rightarrow \mathfrak{F}_i^{\mathfrak{a}}(M) \rightarrow \cdots$$

So, by the inductive hypothesis, $\mathfrak{F}_i^{\mathfrak{a}}((0 :_M x^s)) = 0$ for all $i > t$. Thus $\mathfrak{F}_i^{\mathfrak{a}}(M) = x^s \mathfrak{F}_i^{\mathfrak{a}}(M)$ and so, $\mathfrak{F}_i^{\mathfrak{a}}(M) = 0$ for all $i > t$.

(c) \Rightarrow (a). It is clear. □

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DEPARTMENT OF MATHEMATICS, PAYAME NOOR UNIVERSITY PO BOX 19395-3697, TEHRAN, IRAN

E-mail address: Profbijanzadeh@gmail.com

DEPARTMENT OF MATHEMATICS, PAYAME NOOR UNIVERSITY PO BOX 19395-3697, TEHRAN, IRAN

E-mail address: ghaderi_s@pnu.ac.ir