ROMANIAN JOURNAL OF MATHEMATICS AND COMPUTER SCIENCE, 2015, VOLUME 5, ISSUE 2, p.130-137

FINITENESS PROPERTIES OF FORMAL LOCAL HOMOLOGY MODULES

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Abstract. Let $(R, \mathfrak{m})$ be a commutative Noetherian ring, $\mathfrak{a}$ an ideal of R and M an Artinian R-module. In this paper, we investigate the structure of the formal local homology. We prove several results concerning finiteness properties of formal local homology module.

Mathematics Subject Classification (2010): 13C10, 13D07, 13D05, 16E10, 16E30
Key words: Local homology, Formal local homology, Finitely generated module

Article history:
Received 20 October 2014
Received in revised form 14 July 2015
Accepted 8 August 2015

1. Introduction

Throughout this paper, we assume that $(R, \mathfrak{m})$ is a commutative Noetherian local ring which have non-zero identity. Let $\mathfrak{a}$ be an ideal of R and M an R-module. It is well known that the $\mathfrak{a}$-adic completion functor $\Lambda_{\mathfrak{a}}$ is defined by $\Lambda_{\mathfrak{a}}(M) = \varprojlim_t M/\mathfrak{a}^t M$ (see [8], [9]). In [4], N. T. Coung and T. T. Nam defined the local homology module $H_i^{\mathfrak{a}}(M)$ with respect to $\mathfrak{a}$ by

$$H_i^{\mathfrak{a}}(M) = \varprojlim_n \operatorname{Tor}_i^R(R/a^n, M).$$

If we use $L_i^{\mathfrak{a}}(M)$ to denote the i-th left derived module of $\Lambda_{\mathfrak{a}}(M)$, then $H_i^{\mathfrak{a}}(M) = L_i^{\mathfrak{a}}(M)$ for the Artinian R-modules (see [4]). For an Artinian R-module M, in [2], the i-th formal local homology module of M with respect to $\mathfrak{a}$ is defined by $\varinjlim_n H_i^{\mathfrak{m}}((0 :_M \mathfrak{a}^n))$ and it is investigated its structure. While the formal local cohomology modules are studied in great detail not so much is known about the formal local homology modules (see [1], [5], [7]).

In this paper, for an integer i and an Artinian R-module M, let $\mathfrak{F}_i^{\mathfrak{a}}(M)$ be the formal local homology module of M with respect to $\mathfrak{a}$. Let t be a non-negative integer. It is shown that the local homology module $H_i^{\mathfrak{a}}(M)$ is Artinian for all $i < t$ if and only if there is some non-negative integer s such that $\mathfrak{a}^s H_i^{\mathfrak{a}}(M) = 0$ for all $i < t$, provided that M be Artinian R-module (see [**4**, 4.7]).

We generalize the above result for formal local homology modules. In fact the purpose of this paper is to answer the following question for the formal local homology modules: When are the formal local homology modules finitely generated?

The main aim of this paper is to prove that under some conditions, $\mathfrak{F}_i^{\mathfrak{a}}(M)$ is finitely generated for all $i > t$ if and only if $\mathfrak{a} \subseteq \mathrm{Rad}(\mathrm{Ann}(\mathfrak{F}_i^{\mathfrak{a}}(M)))$ for all $i > t$.

We define the formal finiteness dimension and obtain some finiteness properties of the formal local homology modules. We also show that under some conditions, if $\mathfrak{F}_i^{\mathfrak{a}}(M)$ is finitely generated for all $i > t$ then $\mathrm{Hom}(R/\mathfrak{a}, \mathfrak{F}_t^{\mathfrak{a}}(M))$ is finitely generated.

Throughout this paper, for an R-module M, $\mathrm{E}(R/\mathfrak{m})$ denotes the injective envelope of $R/\mathfrak{m}$ and $\mathrm{D}(M)$ denotes the Matlis duality functor $\mathrm{Hom}(M, \mathrm{E}(R/\mathfrak{m}))$.

2. Main Results

Let $\underline{x} = x_1, \ldots, x_r$ be a system of elements of R, and $b = Rad(\underline{x})$. Let $C^{\cdot}_{\underline{x}}$ denote the $\tilde{C}$ech complex of R with respect to $\underline{x}$, (see [3], [6]). For an R-module M and an ideal $\mathfrak{a}$, the direct system of R-modules $\{(0:_M \mathfrak{a}^n)\}_{n\in N}$ induces a direct system of R-complexes $\{\mathrm{Hom}(C^{\cdot}_{\underline{x}}, (0:_M \mathfrak{a}^n))\}_{n\in N}$. In [2], the formal local homology module is defined by the following isomorphisms

$$H_i(\varinjlim_n \mathrm{Hom}(C^{\cdot}_{\underline{x}}, (0:_M \mathfrak{a}^n))) \simeq \varinjlim_n H_i^b((0:_M \mathfrak{a}^n)),$$

for all $i \in \mathbb{Z}$, provided that M be an Artinian R-module.

Notation. Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$, M be an Antinian R-module and $\mathfrak{m} = Rad(\underline{x})$. We call $\mathfrak{F}_i^{\mathfrak{a}}(M) := \varinjlim_n H_i^{\mathfrak{m}}((0:_M \mathfrak{a}^n))$ the i-th $\mathfrak{a}$-formal local homology.

Definition 2.1. Let M be an Artinian R-module. For an ideal $\mathfrak{a}$ of R, we define the formal finiteness dimension, $ff^{\mathfrak{a}}(M)$ by

$$ff^{\mathfrak{a}}(M) := \inf\{i \mid \mathfrak{F}_i^{\mathfrak{a}}(M) \textit{ is not finitely generated}\}.$$

Proposition 2.2. *Let $\mathfrak{a}$, $\mathfrak{b}$ be two ideals of R and M be an Artinian R-module. If $Rad(\mathfrak{a}) = Rad(\mathfrak{b})$, then $ff^{\mathfrak{a}}(M) = ff^{\mathfrak{b}}(M)$.*

Proof. By assumption $Rad(\mathfrak{a}) = Rad(\mathfrak{b})$, it is easy to prove that if $n \in N$, then there is a positive integer m such that $(0:_M \mathfrak{a}^n) = (0:_M \mathfrak{b}^m)$. So the result follows by the definition of the formal local homology. □

Theorem 2.3. *Let $\mathfrak{a}$ be an ideal of R and M be an Antinian R-module. Then $\mathfrak{F}_i^{\mathfrak{a}}(M) \simeq \mathfrak{F}_i^{\hat{\mathfrak{a}}}(M)$ for all $i \in \mathbb{Z}$, where $\hat{}$ is the completion functor with respect to $\mathfrak{m}$.*

Proof. It is a consequence of theorem [**2**, 2.6] for the particular case $\mathfrak{b} = \mathfrak{m}$. □

Proposition 2.4. *Let $\mathfrak{a}$ be an ideal of R and M be an Antinian R-module. Then $ff^{\mathfrak{a}}(M) = ff^{\hat{\mathfrak{a}}}(M)$, where $\hat{}$ is the completion functor with respect to $\mathfrak{m}$.*

Proof. It immediately follows from Theorem 2.3. □

Theorem 2.5. *Let $\mathfrak{a}$ be an ideal of R. Let M be an Artinian R-module and $x \in \mathfrak{m}$. Then there is the long exact sequence*

$$\cdots \to \mathfrak{F}_i^{(\mathfrak{a},x)}(M) \to \mathfrak{F}_i^{\mathfrak{a}}(M) \to R_x \otimes \mathfrak{F}_i^{\mathfrak{a}}(M) \to \cdots$$

for all $i \in \mathbb{Z}$.

Proof. It is a consequence of theorem [**2**, 2.13] for the particular case $\mathfrak{b} = \mathfrak{m}$. □

Corollary 2.6. *Let x be an element of $\mathfrak{m}$ and M be an Artinian R-module. Then there is a short exact sequence*

$$\cdots \to \varinjlim_{n} H_i^{\mathfrak{m}}((0 :_M x^n)) \to H_i^{\mathfrak{m}}(M) \to R_x \otimes H_i^{\mathfrak{m}}(M) \to \cdots$$

for all $i \in \mathbb{Z}$.

Proof. Whenever $\mathfrak{a} = 0$, it is a consequence of Theorem 2.5. □

Theorem 2.7. *Let $\mathfrak{a}$ be an ideal of R and M be an Antinian R-module. Choose $x \in R$ such that $x \notin \mathfrak{a}$. Then $ff^{\mathfrak{a}}(M) \leq ff^{(\mathfrak{a},x)}(M) + 1$.*

Proof. By Theorem 2.5, there is the following long exact sequence

$$\cdots \to R_x \otimes \mathfrak{F}_{i+1}^{\mathfrak{a}}(M) \to \mathfrak{F}_i^{(\mathfrak{a},x)}(M) \to \mathfrak{F}_i^{\mathfrak{a}}(M) \to \cdots$$

For all $i < ff^{\mathfrak{a}}(M) - 1$, $\mathfrak{F}_{i+1}^{\mathfrak{a}}(M)$ and $\mathfrak{F}_i^{\mathfrak{a}}(M)$ are finitely generated. Therefore $\mathfrak{F}_i^{(\mathfrak{a},x)}(M)$ is finitely generated and this completes the proof. □

Proposition 2.8. *Let M be an Artinian R-module. Then the R-module $H_i^{\mathfrak{m}}(M)$ is finitely generated for all $i \geq 0$.*

Proof. Since $H_i^{\mathfrak{m}}(M) \simeq H_i^{\hat{\mathfrak{m}}}(M)$ as $\hat{R}$-module, so without loss of generality we may assume that R is a complete local ring. By Matlis duality, $\mathrm{D}(M)$ is finitely generated. Therefore $H_{\mathfrak{m}}^i(\mathrm{D}(M))$ is Artinian (see [**3**, 7.1.3]). But by [**4**, 3.3(*ii*)] follows that

$$H_i^{\mathfrak{m}}(M) \simeq H_i^{\mathfrak{m}}(\mathrm{D}(\mathrm{D}(M))) \simeq \mathrm{D}(H_{\mathfrak{m}}^i(\mathrm{D}(M))).$$

Thus $H_i^{\mathfrak{m}}(M)$ is a finitely generated R-module. □

We recall the concept of Krull dimension of an Artinian R-module M, denoted by $\mathrm{Kdim}M$. Let M be an Artinian R-module. When $M = 0$ we put $\mathrm{Kdim}M = -1$. Then by induction, for any ordinal α, we put $\mathrm{Kdim}M = \alpha$ when (i) $\mathrm{Kdim}M < \alpha$ is false, and (ii) for every ascending chain, $M_0 \subseteq M_1 \subseteq \cdots$. of submodules of M, there exists a positive integer m_0 such that $\mathrm{Kdim}(M_{m+1}/M_m) < \alpha$ for all $m \geq m_0$ (see [10]).

For an ideal $\mathfrak{a}$ of $(R, \mathfrak{m})$, we can define the formal homological dimension of M with respect to $\mathfrak{a}$, by

$$f\!-\!cd(\mathfrak{a}, M) = \sup\{i \in \mathbb{Z} : \mathfrak{F}_i^{\mathfrak{a}}(M) \neq 0\}.$$

Theorem 2.9. *Let $\mathfrak{a}$ be an ideal of $(R, \mathfrak{m})$. Then* $\mathrm{Kdim}(0 :_M \mathfrak{a}) = \sup\{i \in \mathbb{Z} : \mathfrak{F}_i^{\mathfrak{a}}(M) \neq 0\}$*, for any non-zero Artinian R-module M.*

Proof. Since $\mathrm{Kdim}(0 :_M \mathfrak{a}^n) = \mathrm{Kdim}(0 :_M \mathfrak{a})$ for all $n \in \mathbb{N}$, by [**4**, 4.8], $\mathfrak{F}_i^{\mathfrak{a}}(M) = 0$ for all $i > \mathrm{Kdim}(0 :_M \mathfrak{a})$. Therefore

$$\mathrm{Kdim}(0 :_M \mathfrak{a}) \geq \sup\{i \in \mathbb{Z} : \mathfrak{F}_i^{\mathfrak{a}}(M) \neq 0\}.$$

In order to prove the equality, put $\mathrm{Kdim}(0 :_M \mathfrak{a}) = s$. First let we consider the short exact sequence

$$0 \to (0 :_M \mathfrak{a}^n) \to (0 :_M \mathfrak{a}^{n+1}) \to \frac{(0 :_M \mathfrak{a}^{n+1})}{(0 :_M \mathfrak{a}^n)} \to 0.$$

Because of $\mathrm{Kdim}((0:_M \mathfrak{a}^{n+1})/(0:_M \mathfrak{a}^n)) < s$, $H^{\mathfrak{m}}_{s+1}((0:_M \mathfrak{a}^{n+1})/(0:_M \mathfrak{a}^n)) = 0$ that induces a monomorphism $0 \to H^{\mathfrak{m}}_s((0:_M \mathfrak{a}^n)) \to H^{\mathfrak{m}}_s((0:_M \mathfrak{a}^{n+1}))$ of non-zero R-modules for all $n \in \mathbb{N}$ (see [**4**, 4.2, 4.10]). So $\mathfrak{F}^{\mathfrak{a}}_s(M) := \varinjlim H^{\mathfrak{m}}_s((0:_M \mathfrak{a}^n)) \neq 0$ and the claim is proved. $\square$

Theorem 2.10. *Let $\mathfrak{a}$ denote an ideal of R. Let M be an Artinian R-module such that* $\mathrm{Kdim}\, M = d$. *Then $\mathfrak{F}^{\mathfrak{a}}_d(M)$ is finitely generated.*

Proof. Let $\mathfrak{a} := (x_1, x_2, \dots, x_n)$. We argue by induction on n. Let $n = 1$. By Corollary 2.6, we have the exact sequence

$$\cdots \to R_{\mathfrak{a}} \otimes H^{\mathfrak{m}}_{d+1}(M) \to \mathfrak{F}^{\mathfrak{a}}_d(M) \xrightarrow{\alpha} H^{\mathfrak{m}}_d(M) \to \cdots$$

By [**4**, 4.8], $H^{\mathfrak{m}}_{d+1}(M) = 0$. So $\mathfrak{F}^{\mathfrak{a}}_d(M) \simeq \mathrm{Im}\alpha \subseteq H^{\mathfrak{m}}_d(M)$ and the claim follows from Proposition 2.8. Now suppose, inductively, that the result has been proved for $n-1$ and let we put $\mathfrak{b} = (x_1, x_2, \dots, x_{n-1})$. By Theorem 2.5, there is a long exact sequence

$$\cdots \to R_{x_n} \otimes \mathfrak{F}^{\mathfrak{b}}_{d+1}(M) \to \mathfrak{F}^{\mathfrak{a}}_d(M) \xrightarrow{\beta} \mathfrak{F}^{\mathfrak{b}}_d(M) \to \cdots$$

So, from Theorem 2.9, $\mathfrak{F}^{\mathfrak{b}}_{d+1}(M) = 0$ and $\mathfrak{F}^{\mathfrak{a}}_d(M) \simeq \mathrm{Im}\beta \subseteq \mathfrak{F}^{\mathfrak{b}}_d(M)$. Therefore the induction hypothesis yields that $\mathfrak{F}^{\mathfrak{a}}_d(M)$ is finitely generated. $\square$

Lemma 2.11. *Let $\mathfrak{a}$ denote an ideal of R, M an Artinian R-module and $N \subset M$ be a summand of M such that* $\mathrm{Supp}(M/N) \subseteq \mathrm{V}(\mathfrak{a})$. *Then there is a natural isomorphism $\mathfrak{F}^{\mathfrak{a}}_i(M/N) \simeq H^{\mathfrak{m}}_i(N)$ for all i and there exists a long exact sequence*

$$\cdots \to H^{\mathfrak{m}}_{i+1}(M/N) \to \mathfrak{F}^{\mathfrak{a}}_i(N) \to \mathfrak{F}^{\mathfrak{a}}_i(M) \to H^{\mathfrak{m}}_i(M/N) \to \cdots$$

Proof. Since $\mathrm{Supp}\,(M/N) \subseteq \mathrm{V}(\mathfrak{a})$, it follows that M/N is annihilated by some power of $\mathfrak{a}$. So

$$\mathfrak{F}^{\mathfrak{a}}_{i+1}(M/N) \simeq \varinjlim_n H^{\mathfrak{m}}_i((0:_{M/N} \mathfrak{a}^n)) \simeq \varinjlim_n H^{\mathfrak{m}}_i(M/N) \simeq H^{\mathfrak{m}}_i(M/N),$$

for all i. By [**2**, 2.9], the short exact sequence $0 \to N \to M \to M/N \to 0$ induces a long exact sequence

$$\cdots \to \mathfrak{F}^{\mathfrak{a}}_{i+1}(M/N) \to \mathfrak{F}^{\mathfrak{a}}_i(N) \to \mathfrak{F}^{\mathfrak{a}}_i(M) \to \mathfrak{F}^{\mathfrak{a}}_i(M/N) \to \cdots$$

and, with the isomorphisms above, the claim is proved for all i. $\square$

Theorem 2.12. *Let M be an Artinian R-module and t be a non-negative integer, such that $\mathfrak{a} \subseteq Rad(Ann(\mathfrak{F}^{\mathfrak{a}}_i(M)))$ for all $i > t$. If $\Lambda_{\mathfrak{a}}(M)$ is projective then $\mathfrak{F}^{\mathfrak{a}}_i(M)$ is finitely generated for all $i > t$.*

Proof. We use induction on $\mathrm{Kdim} M = d$. Let $d = 0$. $\mathfrak{F}^{\mathfrak{a}}_i(M) = 0$ for all $i > 0$, so in this case the claim holds. Now let $d > 0$ and suppose that the claim holds for all value less than d. Since M is Artinian, there exists a positive integer s such that $\mathfrak{a}^t M = \mathfrak{a}^s M$ for all $t \geq s$. So $\bigcap_{t>0} \mathfrak{a}^t M = \mathfrak{a}^s M$, $\Lambda_{\mathfrak{a}}(M) = M/\mathfrak{a}^s M$ and we have a short exact sequence of Artinian R-modules

$$0 \to \bigcap \mathfrak{a}^t M \to M \to \Lambda_{\mathfrak{a}}(M) \to 0.$$

By Lemma 2.11 we get a long exact sequence of local homology modules

$$\cdots \to H^{\mathfrak{m}}_{i+1}(\Lambda_{\mathfrak{a}}(M)) \to \mathfrak{F}^{\mathfrak{a}}_i(\bigcap \mathfrak{a}^t M) \to \mathfrak{F}^{\mathfrak{a}}_i(M) \to H^{\mathfrak{m}}_i(\Lambda_{\mathfrak{a}}(M)) \to \cdots$$

So by Proposition 2.8 it is enough to prove that $\mathfrak{F}^{\mathfrak{a}}_i(\bigcap \mathfrak{a}^t M)$ is finitely generated for all $i > t$. We can replace M by $\bigcap \mathfrak{a}^t M$ and we may assume that $\mathfrak{a}M = M$. Since M is Artinian, $xM = M$ for some $x \in \mathfrak{a}$. Thus, by the hypothesis, there exists a positive integer r such that $x^r \mathfrak{F}^{\mathfrak{a}}_i(M) = 0$ for all $i > t$. Then by [**2**, 2.9] the short exact sequence

$$0 \to (0 :_M x^r) \to M \xrightarrow{x^r} M \to 0$$

induces an exact sequence

$$0 \to \mathfrak{F}^{\mathfrak{a}}_{i+1}(M) \to \mathfrak{F}^{\mathfrak{a}}_i((0 :_M x^r)) \to \mathfrak{F}^{\mathfrak{a}}_i(M) \to 0,$$

for all $i > t$. So, by the inductive hypothesis, $\mathfrak{F}^{\mathfrak{a}}_i((0 :_M x^r))$ is finitely generated for all $i > t$. We see that $\mathfrak{F}^{\mathfrak{a}}_i(M)$ is finitely generated for all $i > t$. This finishes the inductive step. □

Theorem 2.13. *Let $\mathfrak{a}$ be an ideal of R and M be an Artinian R-module. Assume that the integer t is such that $\mathfrak{F}^{\mathfrak{a}}_i(M)$ is finitely generated for all $i > t$. If $\Lambda_{\mathfrak{a}}(M)$ is projective then $(0 :_{\mathfrak{F}^{\mathfrak{a}}_t(M)} \mathfrak{a})$ is finitely generated.*

Proof. We use induction on Kdim$M = n$. For $n = 0$ we have that $\mathfrak{F}^{\mathfrak{a}}_i(M) = 0$ for all $i > 0$ and $\mathfrak{F}^{\mathfrak{a}}_0(M)$ is finitely generated by Theorem 2.10. Now let $n > 0$ and suppose the claim holds for all values less than n. By the same argument as in the proof of Theorem 2.12, there is an element $x \in \mathfrak{a}$, such that $xM = M$. Then the short exact sequence of Artinian modules

$$0 \to (0 :_M x) \to M \xrightarrow{x} M \to 0,$$

implies that

$$\cdots \to \mathfrak{F}^{\mathfrak{a}}_{i+1}(M) \to \mathfrak{F}^{\mathfrak{a}}_i((0 :_M x)) \to \mathfrak{F}^{\mathfrak{a}}_i(M) \to \cdots$$

It yields that $\mathfrak{F}^{\mathfrak{a}}_i((0 :_M x))$ is finitely generated for all $i > t$. Thus, by the induction hypothesis, $(0 :_{\mathfrak{F}^{\mathfrak{a}}_t((0:_M x))} \mathfrak{a})$ is finitely generated. Now consider the exact sequence

$$\cdots \to \mathfrak{F}^{\mathfrak{a}}_{t+1}(M) \xrightarrow{g} \mathfrak{F}^{\mathfrak{a}}_t((0 :_M x)) \xrightarrow{f} \mathfrak{F}^{\mathfrak{a}}_t(M) \xrightarrow{x} \mathfrak{F}^{\mathfrak{a}}_t(M) \to \cdots$$

which breaks into two short exact sequences

$$0 \to \mathrm{Im}g \to \mathfrak{F}^{\mathfrak{a}}_t((0 :_M x)) \to \mathrm{Im}f \to 0,$$

$$0 \to \mathrm{Im}f \to \mathfrak{F}^{\mathfrak{a}}_t(M) \xrightarrow{x} \mathfrak{F}^{\mathfrak{a}}_t(M).$$

The first of these sequences induces a long exact sequence

$$\cdots \to (0 :_{\mathfrak{F}^{\mathfrak{a}}_t((0:_M x))} \mathfrak{a}) \to (0 :_{\mathrm{Im}f} \mathfrak{a}) \to \mathrm{Ext}^1(R/\mathfrak{a}, \mathrm{Im}g) \cdots$$

where $(0 :_{\mathfrak{F}^{\mathfrak{a}}_t((0:_M x))})$ and $\mathrm{Ext}^1(R/\mathfrak{a}, \mathrm{Im}g)$ are finitely generated. The second sequence induces a long exact sequence

$$0 \to (0 :_{\mathrm{Im}f} \mathfrak{a}) \to (0 :_{\mathfrak{F}^{\mathfrak{a}}_t(M)} \mathfrak{a}) \xrightarrow{x} (0 :_{\mathfrak{F}^{\mathfrak{a}}_t(M)} \mathfrak{a}) \cdots$$

Since $x \in \mathfrak{a}$, it follows that $(0 :_{\mathrm{Im}f} \mathfrak{a}) \simeq (0 :_{\mathfrak{F}^{\mathfrak{a}}_t(M)} \mathfrak{a})$. This completes the proof. □

Corollary 2.14. *Let M be an Artinian R-module. Then $(0 :_{\mathfrak{F}^{\mathfrak{a}}_{f-cd(\mathfrak{a},M)}(M)} \mathfrak{a})$ is finitely generated.*

Proof. It immediately follows from Theorem 2.13. $\square$

Theorem 2.15. *Let $\mathfrak{a}$ be an ideal of R and M be an Artinian R-module. Let t be a non-negative integer. Then the following statements are equivalent:*

(*a*) $\mathfrak{m} \subseteq \mathrm{Rad}(\mathrm{Ann}(\mathfrak{F}^{\mathfrak{a}}_i(M)))$ *for all* $i > t$,
(*b*) $\mathfrak{F}^{\mathfrak{a}}_i(M)$ *is finitely generated for all* $i > t$,
(*c*) $\mathfrak{F}^{\mathfrak{a}}_i(M) = 0$ *for all* $i > t$.

Proof. $(a) \Rightarrow (b)$. We use induction on KdimM $= d$. Let $d = 0$. $\mathfrak{F}^{\mathfrak{a}}_i(M) = 0$ for all $i > 0$, so in this case the claim holds. Now let $d > 0$ and suppose that the claim hold for all value less than d. Since $(0 :_M \mathfrak{a}^n)$ is Artinian, we can replace it by $\bigcap_{t>0} \mathfrak{m}^t(0 :_M \mathfrak{a}^n)$ (see [**4**, 4.5]). We can assume that $\mathfrak{m}(0 :_M \mathfrak{a}^n) = (0 :_M \mathfrak{a}^n)$ and since M is Artinian, $x(0 :_M \mathfrak{a}^n) = (0 :_M \mathfrak{a}^n)$ for some $x \in \mathfrak{m}$. From the hypothesis it follows that there exists a positive integer s such that $x^s\mathfrak{F}^{\mathfrak{a}}_i(M) = 0$ for all $i > t$. Thus for all $i > t$ the exact sequence

$$0 \to (0 :_{(0:_M x^s)} \mathfrak{a}^n) \to (0 :_M \mathfrak{a}^n) \xrightarrow{x^s} (0 :_M \mathfrak{a}^n) \to 0,$$

implies that

$$0 \to \mathfrak{F}^{\mathfrak{a}}_{i+1}(M) \to \mathfrak{F}^{\mathfrak{a}}_i((0 :_M x^s)) \to \mathfrak{F}^{\mathfrak{a}}_i(M) \to 0.$$

So, by the inductive hypothesis, $\mathfrak{F}^{\mathfrak{a}}_i((0 :_M x^s))$ is finitely generated for all $i > t$, and we see that $\mathfrak{F}^{\mathfrak{a}}_i(M)$ is finitely generated for all $i > t$.

$(b) \Rightarrow (c)$. We use induction on KdimM $= d$. Let $d = 0$. $\mathfrak{F}^{\mathfrak{a}}_i(M) = 0$ for all $i > 0$, so in this case the claim holds. Now let $d > 0$ and suppose that the claim hold for all value less than d. As we did in the proof of $(a) \Rightarrow (b)$, there is the following exact sequence

$$\cdots \to \mathfrak{F}^{\mathfrak{a}}_{i+1}(M) \to \mathfrak{F}^{\mathfrak{a}}_i((0 :_M x^s)) \to \mathfrak{F}^{\mathfrak{a}}_i(M) \to \cdots$$

So, by the inductive hypothesis, $\mathfrak{F}^{\mathfrak{a}}_i((0 :_M x^s)) = 0$ for all $i > t$. Thus $\mathfrak{F}^{\mathfrak{a}}_i(M) = x^s\mathfrak{F}^{\mathfrak{a}}_i(M)$ and so, $\mathfrak{F}^{\mathfrak{a}}_i(M) = 0$ for all $i > t$.

$(c) \Rightarrow (a)$. It is clear.

$\square$

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